

Initial and boundary blow-up problem for p -Laplacian parabolic equation with general absorption ¹

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Abstract. In this article, we investigate the initial and boundary blow-up problem for the p -Laplacian parabolic equation $u_t - \Delta_p u = -b(x, t)f(u)$ over a smooth bounded domain Ω of \mathbb{R}^N with $N \geq 2$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $p > 1$, and $f(u)$ is a function of regular variation at infinity. We study the existence and uniqueness of positive solutions, and their asymptotic behaviors near the parabolic boundary.

Key words: p -Laplacian parabolic equation; Initial and boundary blow-up; Positive solutions; Asymptotic behaviors.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open bounded domain with smooth boundary $\partial\Omega$, and $\Omega_T := \Omega \times (0, T)$ with $0 < T < \infty$. The aim of this paper is to study the p -Laplacian parabolic equation

$$u_t - \Delta_p u = -b(x, t)f(u), \quad (x, t) \in \Omega_T, \quad (1.1)$$

with blow-up initial and boundary values:

$$u = \infty, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.2)$$

$$u = \infty, \quad (x, t) \in \bar{\Omega} \times \{0\}, \quad (1.3)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $p > 1$, $b(x, t)$ is a positive continuous function in Ω_T ($b(x, T) = 0$ or $b(x, T) = \infty$ is allowed), and $f \in C^1([0, \infty))$ with $f(0) = 0$ and $f'(u) > 0$ for $u > 0$.

Throughout this work, by (1.2)-(1.3), we mean that

$$\begin{cases} u(x, t) \rightarrow \infty & \text{as } d(x) \rightarrow 0 \text{ uniformly for } t \in (0, T), \\ u(x, t) \rightarrow \infty & \text{as } t \rightarrow 0 \text{ uniformly for } x \in \bar{\Omega}, \end{cases}$$

where, unless specified otherwise, $d(x) = d(x, \partial\Omega) = \operatorname{dist}(x, \partial\Omega)$ represents the distance from x to $\partial\Omega$ for $x \in \Omega$.

Remark 1.1 *The results of this paper remain valid if we add the term $a(x, t)u^{p-1}$ to the right hand side of the equation (1.1). For simplicity, we have not included this term.*

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We are interested in the existence and uniqueness of positive weak solutions to (1.1)–(1.3), and the behavior of the solutions near the parabolic boundary

$$\Sigma_T := \partial\Omega \times (0, T) \cup \bar{\Omega} \times \{0\}.$$

While there is an abundance of work – going back to Bieberbach in 1916 – on boundary blow-up for elliptic equations, the corresponding investigation for parabolic equations has lagged behind. In 1994, Bandle et. al. [3] studied the existence, uniqueness and asymptotic behavior near the parabolic boundary of solutions to the autonomous parabolic boundary blow-up problem

$$\begin{cases} u_t - \Delta\phi(u) = -f(u), & (x, t) \in \Omega \times (0, \infty), \\ u = \infty, & (x, t) \in \partial\Omega \times (0, \infty) \cup \bar{\Omega} \times \{0\}. \end{cases} \quad (1.4)$$

In particular, they proved that, under suitable conditions on the functions ϕ and f ,

$$\begin{aligned} \frac{u(x, t)}{w(t)} &\rightarrow 1 \quad \text{as } (x, t) \rightarrow \Omega \times \{0\}, \\ \frac{u(x, t)}{V(x)} &\rightarrow 1 \quad \text{as } (x, t) \rightarrow \partial\Omega \times (0, \infty), \end{aligned}$$

where $w(t)$ is a solution of

$$w' = -f(w), \quad t > 0; \quad w(0) = \infty, \quad (1.5)$$

and $V(x)$ is the unique solution to the elliptic boundary blow-up problem

$$\Delta\phi(v) = f(v), \quad x \in \Omega; \quad v|_{\partial\Omega} = \infty. \quad (1.6)$$

In [17], Marcus and Véron showed that if f is super-additive, i.e.,

$$f(u + v) \geq f(u) + f(v), \quad \forall u, v \geq 0,$$

and satisfies

$$\int_a^\infty \frac{ds}{f(s)} < \infty, \quad \int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty,$$

where a is a non-negative constant such that $f(u)$ is positive and continuous when $u > a$ and $F(s) = \int_0^s f(\tau)d\tau$, then there exists a maximal solution $\bar{u}(x, t)$ to (1.4), and

$$\bar{u}(x, t) \leq w(t) + V(x), \quad \bar{u}(x, t) \geq \max\{w(t), V(x)\}, \quad \forall (x, t) \in \Omega \times (0, \infty).$$

For the non-autonomous case, very recently, motivated by a spatial-temporal degeneracy problem for the diffusive logistic equation used in population dynamics, Du et. al. [12] investigated the following problem:

$$\begin{cases} u_t - \Delta u = a(x, t)u - b(x, t)u^q, & (x, t) \in \Omega \times (0, T), \\ u = \infty, & (x, t) \in \partial\Omega \times (0, T) \cup \bar{\Omega} \times \{0\}, \end{cases}$$

where $q > 1$, $a(x, t)$ and $b(x, t)$ are continuous functions in $\bar{\Omega} \times [0, T]$ and $\Omega \times [0, T]$, respectively, and $b(x, t)$ satisfies

$$\alpha_1(t)d^\beta(x) \leq b(x, t) \leq \alpha_2(t)d^\beta(x), \quad \forall (x, t) \in \Omega \times [0, T]$$

with $\beta > -2$, and $\alpha_1(t)$ and $\alpha_2(t)$ being positive continuous functions in $[0, T)$. They also obtained existence, uniqueness and asymptotic behavior results. Furthermore, under the extra condition that $b(x, t) \geq c(T - t)^\theta d^\beta(x)$ for some constants $c > 0, \theta > 0$ and $\beta > -2$, they showed that the positive solution that exists stays bounded in any compact subset of Ω as t increases to T , and hence solves the equation up to $t = T$.

Related problems have also been studied by [1, 2, 5, 14] and [20]. Especially, in [14] the authors proved the existence of large solutions for the problems

$$\begin{cases} u_t - \operatorname{div} a(x, t, u, \nabla u) + g(x, t, u, \nabla u) = f(x, t), & (x, t) \in \Omega \times (0, T), \\ u = u_0, & (x, t) \in \Omega \times \{0\}, \\ u = \infty, & (x, t) \in \partial\Omega \times (0, T), \end{cases}$$

where $\operatorname{div} a(x, t, u, \nabla u) \approx \Delta_p u$, $g(x, t, u, \nabla u) \approx u|\nabla u|^q$ with $p - 1 < q \leq p$, and $u_0 \in L^1_{\text{loc}}(\Omega)$, $f \in L^1(0, T; L^1_{\text{loc}}(\Omega))$ with $f^- \in L^1((0, T) \times \Omega)$. In [20], the existence and uniqueness of entropy large solutions was discussed for the following problem

$$\begin{cases} u_t - \Delta_p u = 0, & (x, t) \in \Omega \times (0, T), \\ u = u_0, & (x, t) \in \Omega \times \{0\}, \\ u = \infty, & (x, t) \in \partial\Omega \times (0, T), \end{cases}$$

where $1 \leq p < 2$, $u_0 \in L^1_{\text{loc}}(\Omega)$ ($u_0 \in L^1(\Omega)$ if $p = 1$) is a nonnegative function.

Motivated by the above works, in this paper, we study the problem (1.1)–(1.3). We are able to extend some of the results of [3, 12, 17]. Our method refers to Karamata's regular variation theory [4], which has been used by many authors in elliptic boundary blow-up problems.

We briefly recall some key notions of Karamata's theory; more can be found in the Appendix.

A measurable function $R : [A, \infty) \rightarrow (0, \infty)$, for some $A > 0$, is called *regularly varying* at infinity of index $\rho \in \mathbb{R}$, for short $R \in RV_\rho$, if $\lim_{u \rightarrow \infty} \frac{R(\xi u)}{R(u)} = \xi^\rho$, $\forall \xi > 0$. When the index ρ is zero, we call the function R *slowly varying* at infinity.

Following [8] (see also [19]), we denote by \mathcal{K}_ℓ the set of all positive, monotonic functions $k \in C^1(0, \mu) \cap L^1(0, \mu)$ that satisfy

$$\lim_{s \rightarrow 0^+} \left(\frac{K(s)}{k(s)} \right)' = \ell \in (0, \infty),$$

where $K(s) = \int_0^s k(\theta) d\theta$ and $\mu \geq \operatorname{diam}(\Omega)$. For any $k \in \mathcal{K}_\ell$, it is clear that $\lim_{s \rightarrow 0^+} \frac{K(s)}{k(s)} = 0$ and $\lim_{s \rightarrow 0^+} \frac{K(s)k'(s)}{k^2(s)} = 1 - \ell$. Moreover, $0 < \ell \leq 1$ if k is non-decreasing, and $\ell \geq 1$ if k is non-increasing.

With regard to (1.1), we shall often make the following assumptions:

(F₁) $f \in RV_\rho$ with $\rho > p - 1$;

(F₂) The function $s \mapsto s^{-(p-1)} f(s)$ is increasing in $(0, \infty)$;

(B) There exist a function $k \in \mathcal{K}_\ell$ and two positive continuous functions $\alpha_1(t)$ and $\alpha_2(t)$ defined on $[0, T)$, such that

$$\alpha_1(t)k^p(d(x)) \leq b(x, t) \leq \alpha_2(t)k^p(d(x)), \quad \forall (x, t) \in \Omega_T,$$

where $\alpha_1(T) = 0$ or $\alpha_2(T) = \infty$ may occur.

Remark 1.2 *If we assume that both $\alpha_1(t)$ and $\alpha_2(t)$ are positive and continuous on $[0, T]$, then one can replace Ω_T by $Q_T := \Omega \times (0, T]$ and the problem can be discussed in Q_T . Moreover, Theorem 1.1 below will then also hold true for $t^* = T$.*

For notation, let ϕ be the function defined uniquely by

$$\int_{\phi(t)}^{\infty} \frac{ds}{(p'F(s))^{1/p}} = t, \quad t > 0, \quad (1.7)$$

where $F(t) = \int_0^t f(s)ds$ and $p' = \frac{p}{p-1}$. It is easily seen that $\phi(0) = \infty$. Further, let $\xi(t)$ be the unique positive solution of (1.5) and $\xi^*(t)$ be the unique positive solution of

$$(\xi^*)' = -f^*(\xi^*), \quad t > 0; \quad \xi^*(0) = \infty, \quad (1.8)$$

with $f^*(s) = (k \circ K^{-1} \circ \phi^{-1}(s))^p f(s)$.

Theorem 1.1 (i) *Let the conditions (F_1) , (F_2) and (B) hold. Suppose that*

$$\rho > \max \{1, p-1, p-1-(p-2)/\ell\}.$$

Then the problem (1.1)-(1.3) has a maximal positive solution \bar{u} and a minimal positive solution \underline{u} , in the sense that any positive solution u of (1.1)-(1.3) satisfies $\underline{u} \leq u \leq \bar{u}$. Moreover, the minimal positive solution \underline{u} is non-increasing in t . Furthermore, for any given $t^ \in (0, T)$, there is a constant $C > 0$, depending on t^* , such that the maximal positive solution \bar{u} satisfies*

$$\bar{u}(x, t) \leq \begin{cases} C [\xi(t) + \phi(K(d(x)))], & \text{if } k \text{ is non-increasing,} \\ C [\xi^*(t) + \phi(K(d(x)))], & \text{if } k \text{ is non-decreasing,} \end{cases} \quad (1.9)$$

for all $(x, t) \in \Omega \times (0, t^]$.*

(ii) *Assume that in addition f satisfies the following condition:*

(C) *There is a constant $l > \max\{1, p-1\}$ such that $f(u) \geq \varepsilon^{-l} f(\varepsilon u)$ for all $u > 0$ and $0 < \varepsilon \ll 1$.*

Then, for any given $t^ \in (0, T)$, there is a constant $c > 0$, depending on t^* , such that the minimal positive solution \underline{u} satisfies*

$$\underline{u}(x, t) \geq \begin{cases} c [\xi^*(t) + \phi(K(d(x)))], & \text{if } k \text{ is non-increasing,} \\ c [\xi(t) + \phi(K(d(x)))], & \text{if } k \text{ is non-decreasing,} \end{cases} \quad (1.10)$$

for all $(x, t) \in \Omega \times (0, t^]$.*

Remark 1.3 (i) *If there is a constant $l > \max\{1, p-1\}$ such that the function $f(u)/u^l$ is increasing for $u > 0$, then the condition (C) holds.*

(ii) *Set $q = \rho - (\rho - p + 1)(1 - \ell)$. Under the conditions of Theorem 1.1, we have $f^* \in RV_q$ (see (A.2) below).*

(iii) *Clearly, $\rho > q$ if $0 < \ell < 1$; $\rho = q$ if $\ell = 1$; $\rho < q$ if $\ell > 1$. As $\rho > \max\{1, p-1, p-1-(p-2)/\ell\}$, we have $q > \max\{1, p-1\}$.*

To simplify notation, we denote

$$r = \frac{\rho + 1}{\rho + 1 - p}.$$

Theorem 1.2 *Under the assumptions of Theorem 1.1, let $u(x, t)$ be any positive solution of (1.1)-(1.3). Then the following hold:*

(i) *For any fixed $t_0 \in (0, T)$ and $y \in \partial\Omega$, we have*

$$\lim_{\Omega \ni x \rightarrow y} \frac{u(x, t_0)}{\phi(K(d(x)))} = \left(\frac{r + \ell - 1}{r\beta(y, t_0)} \right)^{\frac{r-1}{p}}$$

provided that $\beta(x, t) := \frac{b(x, t)}{k^p(d(x))}$ can be extended to a continuous function on $\bar{\Omega} \times (0, T)$.

(ii) *For any fixed $x_0 \in \Omega$, let $\tau(t)$ be the unique positive solution of*

$$\tau' = -b(x_0, 0)f(\tau), \quad t > 0; \quad \tau(0) = \infty. \quad (1.11)$$

Then

$$\limsup_{t \rightarrow 0} \frac{u(x_0, t)}{\tau(t)} \leq 1. \quad (1.12)$$

If in addition $p > 2N/(N + 2)$ and $f(s)/s$ is increasing for $s > 0$, then

$$\liminf_{t \rightarrow 0} \frac{u(x_0, t)}{\tau(t)} \geq 1. \quad (1.13)$$

Theorem 1.3 *Under the assumptions of Theorem 1.1, if $p = 2$ and $k(s) = 1$, and $f(u)$ is convex in $(0, \infty)$, then (1.1)-(1.3) has a unique positive solution.*

This paper is organized as follows. In Section 2, we prove the comparison principle. Section 3 is devoted to prove Theorem 1.1. The proofs of asymptotic behavior and uniqueness (Theorems 1.2 and 1.3) will be given in Section 4. The last section (Appendix) contains three parts: (i) state and prove some relevant results of the Karamata's regular variation theory which will be used in the text (not all of which are readily available in the literature). Especially, Lemmas A.5-A.8 play an important role in the proofs of Theorem 1.1 and Theorem 1.2(i); (ii) prove some results on the unique solution of (1.5), which will be used in the proof of Theorem 1.2(ii); (iii) state some results on the corresponding elliptic boundary blow-up problem.

2 Preliminaries

The main aim of this section is to prove the key comparison principle that is crucial to this paper. While the comparison principle is, in a sense, known, we believe a careful proof is useful to clarify the different versions that appear in the literature.

We first establish a notation: If $\varphi \in C^\infty(\Omega_T)$ and $\text{supp } \varphi \subset \subset \Omega_T$, i.e. φ is zero near the parabolic boundary $\Sigma_T := \partial\Omega \times (0, T) \cup \bar{\Omega} \times \{0\}$ of Ω_T , we write $\varphi \in C_\bullet^\infty(\Omega_T)$.

Definition 2.1 *A weak lower (upper) solution of the equation (1.1) is a measurable function $u(x, t)$ such that*

$$u \in C(t_0, T; L^2(\Omega')) \cap L^p(t_0, T; W^{1,p}(\Omega')) \cap L^\infty(\Omega' \times (t_0, T)), \quad u_t \in L^2(\Omega' \times (t_0, T))$$

for any $0 < t_0 < T$ and any compact subset Ω' of Ω ; and

$$\int_{\Omega} u \varphi dx + \int_0^t \int_{\Omega} \{ |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + b f(u) \varphi \} dx d\tau \leq (\geq) \int_0^t \int_{\Omega} u \varphi_t dx d\tau, \quad \forall 0 < t < T. \quad (2.1)$$

for all test function $\varphi \in C_{\bullet}^{\infty}(\Omega_T)$, $\varphi \geq 0$ in Ω_T .

A function u that is both a lower solution and a upper solution is a weak solution of the equation (1.1).

Proposition 2.1 (Comparison Principle) *Let $f \in C[0, \infty)$ be a non-negative function, and $b(x, t) \in C(\Omega_T)$ be a non-negative and non-trivial function. Assume that $u_1, u_2 \in C^1(\Omega_T)$ are weak upper and lower solutions of equation (1.1) respectively, that are positive in Ω_T . If $f(s)$ is non-decreasing for $s \in (\inf_{\Omega_T} \{u_1, u_2\}, \sup_{\Omega_T} \{u_1, u_2\})$, and u_1, u_2 satisfy*

$$\limsup_{(x,t) \rightarrow \Sigma_T} (u_2 - u_1) \leq 0, \quad (2.2)$$

then $u_1 \geq u_2$ in Ω_T .

Proof. The proof refers to the corresponding elliptic case ([11]), and makes use of [9, Lemma 2.1]. Let $\varphi \in C_{\bullet}^{\infty}(\Omega_T)$ be a non-negative function. Then we have

$$\begin{aligned} & \int_{\Omega} (u_2 - u_1) \varphi dx + \int_0^t \int_{\Omega} (|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1) \cdot \nabla \varphi dx d\tau \\ & + \int_0^t \int_{\Omega} b[f(u_2) - f(u_1)] \varphi dx d\tau \leq \int_0^t \int_{\Omega} (u_2 - u_1) \varphi_t dx d\tau, \quad \forall 0 < t < T. \end{aligned} \quad (2.3)$$

For any $0 < \varepsilon < 1$, let $v = [u_2 - (u_1 + \varepsilon)]_+$ where $u_+ := \max\{u, 0\}$. By the assumption (2.2),

$$\limsup_{(x,t) \rightarrow \Sigma_T} (u_2 - u_1 - \varepsilon) \leq \limsup_{(x,t) \rightarrow \Sigma_T} (u_2 - u_1) - \varepsilon \leq -\varepsilon.$$

We can choose $t_{\varepsilon} \in (0, T)$ and $\Omega(\varepsilon) \subset\subset \Omega$ with $t_{\varepsilon} \rightarrow 0$ and $\Omega(\varepsilon) \rightarrow \Omega$ as $\varepsilon \rightarrow 0$, such that $v = 0$ in $\Omega_T \setminus \Omega(\varepsilon) \times (2t_{\varepsilon}, T)$ and

$$v \in W^{1,2}(t_{\varepsilon}, T; L^2(\Omega(\varepsilon))) \cap L^p(t_{\varepsilon}, T; W^{1,p}(\Omega(\varepsilon))) \cap L^{\infty}(\Omega(\varepsilon) \times (t_{\varepsilon}, T)).$$

It follows that v can be approximated arbitrarily closely in the norm of

$$W^{1,2}(t_{\varepsilon}, T; L^2(\Omega(\varepsilon))) \cap L^p(t_{\varepsilon}, T; W^{1,p}(\Omega(\varepsilon))) \cap L^{\infty}(\Omega(\varepsilon) \times (t_{\varepsilon}, T))$$

by $C_{\bullet}^{\infty}(\Omega_T)$ functions. Thus (2.3) holds with φ replaced by v . For any given $t_{\varepsilon} < s < T$, denote

$$D_s^{\varepsilon} = \{(x, t) \in \Omega(\varepsilon) \times (t_{\varepsilon}, s] : u_2(x, t) > u_1(x, t) + \varepsilon\}, \quad C_s^{\varepsilon} = \{(x, t) \in D_s^{\varepsilon} : t = s\}.$$

To simplify the notation we write $w = u_2 - u_1$. Then for any fixed $t_{\varepsilon} < t < T$, we have

$$\begin{aligned} & \int_{C_t^{\varepsilon}} w(w - \varepsilon)_+ dx + \int_{D_t^{\varepsilon}} (|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1) \cdot \nabla (u_2 - u_1) dx \\ & + \int_{D_t^{\varepsilon}} b(f(u_2) - f(u_1)) (w - \varepsilon)_+ dx d\tau \leq \int_{D_t^{\varepsilon}} w[(w - \varepsilon)_+]_t dx d\tau. \end{aligned} \quad (2.4)$$

It is obviously that the third term in the left hand side of (2.4) is non-negative since $u_2 > u_1$ in D_t^ε and f is non-decreasing and b is positive. By [9, Lemma 2.1], we see that the second term in the left hand side of (2.4) is also non-negative. Therefore

$$\int_{C_t^\varepsilon} w(w - \varepsilon)_+ dx \leq \int_{D_t^\varepsilon} w[(w - \varepsilon)_+]_t dx d\tau. \quad (2.5)$$

Noting that $C_t^\varepsilon \subset \Omega(\varepsilon) \times \{t\}$ and $w(x, t) \leq \varepsilon$ in $(\Omega(\varepsilon) \times \{t\}) \setminus C_t^\varepsilon$, and $D_t^\varepsilon \subset \Omega(\varepsilon) \times (t_\varepsilon, t]$ and $w(x, \tau) \leq \varepsilon$ in $\Omega(\varepsilon) \times (t_\varepsilon, t] \setminus D_t^\varepsilon$, we have

$$\begin{aligned} \int_{C_t^\varepsilon} w(w - \varepsilon)_+ dx &= \int_{C_t^\varepsilon} (w - \varepsilon)_+^2 dx + \varepsilon \int_{C_t^\varepsilon} (w - \varepsilon)_+ dx \\ &= \int_{\Omega(\varepsilon)} (w - \varepsilon)_+^2 dx + \varepsilon \int_{\Omega(\varepsilon)} (w - \varepsilon)_+ dx, \\ \int_{D_t^\varepsilon} w[(w - \varepsilon)_+]_t dx d\tau &= \frac{1}{2} \int_{t_\varepsilon}^t \int_{\Omega(\varepsilon)} [(w - \varepsilon)_+^2]_t dx d\tau + \varepsilon \int_{t_\varepsilon}^t \int_{\Omega(\varepsilon)} [(w - \varepsilon)_+]_t dx d\tau \\ &= \frac{1}{2} \int_{\Omega(\varepsilon)} (w - \varepsilon)_+^2 dx + \varepsilon \int_{\Omega(\varepsilon)} (w - \varepsilon)_+ dx. \end{aligned}$$

This combined with (2.5) yields

$$\int_{\Omega(\varepsilon)} (w(x, t) - \varepsilon)_+^2 dx = 0, \quad \forall t_\varepsilon < t < T,$$

which implies that $w(x, t) \leq \varepsilon$, i.e., $u_2(x, t) \leq u_1(x, t) + \varepsilon$ in $\Omega(\varepsilon) \times (t_\varepsilon, T]$. Noting that $t_\varepsilon \rightarrow 0$ and $\Omega(\varepsilon) \rightarrow \Omega$ as $\varepsilon \rightarrow 0^+$, we conclude $u_2(x, t) \leq u_1(x, t)$ a.e. in Ω_T and complete the proof. \square

For ease of reference, we end this section by recalling the following comparison principle for the corresponding elliptic problem, which can be derived from the characterizations of the maximum principle in [13] and Proposition 2.2 in [11, 15].

Proposition 2.2 (Comparison Principle) *Suppose that D is a bounded domain in \mathbb{R}^N , and $\beta(x)$ is a continuous function in D with $\beta(x) \geq 0, \beta(x) \not\equiv 0$. Let $u_1, u_2 \in C^1(D)$ be positive in D and satisfy in the sense of distribution*

$$-\Delta_p u_1 + \beta(x)g(u_1) \geq 0 \geq -\Delta_p u_2 + \beta(x)g(u_2)$$

and

$$\limsup_{d(x, \partial D) \rightarrow 0} (u_2 - u_1) \leq 0,$$

where $g \in C([0, \infty), [0, \infty))$. If furthermore we assume that $g(s)/s^{p-1}$ is increasing for $s \in (\inf_D \{u_1, u_2\}, \sup_D \{u_1, u_2\})$, then $u_1 \geq u_2$ in D .

3 Maximal and minimal positive solutions

In this section, we give the proof of Theorem 1.1. We shall divide the proof into five steps. Some of the techniques used are based on those found in [12] and [6], though the adaptation to our setting is not straightforward.

Step 1: Construction of upper solution.

This is the key step in the proof. By Theorem A.2(i), the problem

$$\begin{cases} \Delta_p z = k^p(d(x))f(z), & x \in \Omega, \\ z = \infty, & x \in \partial\Omega \end{cases} \quad (3.1)$$

has a unique positive solution $z(x)$, and there are positive constants c_1 and c_2 such that

$$c_1\phi(K(d(x))) \leq z(x) \leq c_2\phi(K(d(x))), \quad x \in \Omega. \quad (3.2)$$

Case (1): k is non-increasing.

For arbitrarily small $\varepsilon > 0$, since $\alpha_1(t) > 0$ in $[0, T - \varepsilon]$, we may assume that $\alpha_1(t) \geq \alpha_\varepsilon$ on $[0, T - \varepsilon]$ for some constant $\alpha_\varepsilon > 0$. Let $\xi(t)$ be the unique positive solution of (1.5). By the assumption on k , we can find $c > 0$ such that $ck^p(d(x)) \geq 1$ in Ω . It follows that $\xi'(t) \geq -ck^p(d(x))f(\xi(t))$ in Ω_T . By Lemma A.5, we can find $\Lambda > 1$ sufficiently large such that $(c\Lambda + \Lambda^{p-1})f(\xi + z) < \alpha_\varepsilon f(\Lambda\xi + \Lambda z)$. Since f is increasing, the function $\bar{\mathbf{u}}(x, t) = \Lambda[\xi(t) + z(x)]$ satisfies (here, $d = d(x)$)

$$\begin{aligned} \bar{\mathbf{u}}_t - \Delta_p \bar{\mathbf{u}} &\geq -c\Lambda k^p(d)f(\xi) - \Lambda^{p-1}k^p(d)f(z) \\ &\geq -c\Lambda k^p(d)f(\xi + z) - \Lambda^{p-1}k^p(d)f(\xi + z) \\ &\geq -\alpha_\varepsilon k^p(d)f(\bar{\mathbf{u}}) \\ &\geq -b(x, t)f(\bar{\mathbf{u}}), \quad (x, t) \in \Omega \times (0, T - \varepsilon]. \end{aligned} \quad (3.3)$$

Case (2): k is non-decreasing.

Let c_1 and c_2 be given by (3.2). Noting that ϕ is decreasing, K is increasing and k is non-decreasing, it follows that

$$\begin{cases} k(d) \geq k \circ K^{-1} \circ \phi^{-1}(c_1^{-1}s) & \text{if } \phi(K(d)) \leq c_1^{-1}s, \\ k(d) \leq k \circ K^{-1} \circ \phi^{-1}(c_2^{-1}s) & \text{if } \phi(K(d)) \geq c_2^{-1}s. \end{cases}$$

Set $f_i(s) = (k \circ K^{-1} \circ \phi^{-1}(c_i^{-1}s))^p f(s)$. Then $f_i(s) \in RV_q$ by (A.2), and

$$f_1(s) \leq k^p(d)f(s) \text{ when } s \geq c_1\phi(K(d)), \quad f_2(s) \geq k^p(d)f(s) \text{ when } s \leq c_2\phi(K(d)). \quad (3.4)$$

By virtue of (A.1), it can be deduced that

$$\lim_{s \rightarrow \infty} \frac{f_2(s)}{f_1(s)} = \lim_{s \rightarrow \infty} \frac{(k \circ K^{-1} \circ \phi^{-1}(c_2^{-1}s))^p}{(k \circ K^{-1} \circ \phi^{-1}(c_1^{-1}s))^p} = \left(\frac{c_1}{c_2}\right)^{p\frac{1-\ell}{1-r}}.$$

There is a constant $s_0 > 0$ such that

$$\frac{1}{2}(c_1/c_2)^{p\frac{1-\ell}{1-r}} f_1(s) \leq f_2(s) \leq 2(c_1/c_2)^{p\frac{1-\ell}{1-r}} f_1(s), \quad \forall s \geq s_0. \quad (3.5)$$

Let $v(t)$ be the unique positive solution of

$$v' = -f_2(v), \quad t > 0; \quad v(0) = \infty, \quad (3.6)$$

and take $\tau = \min\{v(T), c_1 \inf_{\Omega} \phi(K(d(x)))\} > 0$. Since $f_1(s)$ and $f_2(s)$ are positive and continuous in $(0, \infty)$, there are positive constants C_i such that $C_1 f_1(s) \leq f_2(s) \leq C_2 f_1(s)$ for all $\tau \leq s \leq s_0$. This combined with (3.5) yields the existence of a positive constant C such that

$$C^{-1}f_1(s) \leq f_2(s) \leq Cf_1(s), \quad \forall s \geq \tau. \quad (3.7)$$

Since $q > 0$, by Lemma A.8, there are a positive, continuous and increasing function $g \in RV_q$ and a constant $\sigma > 0$ such that $\sigma g(s) \leq f_2(s) \leq g(s)$ for all $s \geq \tau$. Hence, by (3.7)

$$C^{-1}f_1(s) \leq g(s) \leq Cf_2(s), \quad \forall s \geq \tau. \quad (3.8)$$

Let $\Lambda > 0$ be a constant and $\bar{\mathbf{u}}(x, t) = \Lambda[v(t) + z(x)]$. Then we have

$$\bar{\mathbf{u}}_t - \Delta_p \bar{\mathbf{u}} = -\Lambda f_2(v) - \Lambda^{p-1} k^p(d(x))f(z).$$

By (3.2), we have $c_2^{-1}z(x) \leq \phi(K(d(x))) \leq c_1^{-1}z(x)$, which implies $k^p(d(x))f(z(x)) \leq f_2(z(x))$. It follows from (3.7) and (3.8) that

$$\begin{aligned} \Lambda f_2(v) + \Lambda^{p-1} k^p(d)f(z) &\leq \Lambda f_2(v) + \Lambda^{p-1} f_2(z) \\ &\leq (\Lambda + \Lambda^{p-1}) g(v + z) \\ &\leq (\Lambda + \Lambda^{p-1}) Cf_2(v + z) \\ &\leq (\Lambda + \Lambda^{p-1}) Cf_1(v + z). \end{aligned}$$

Since $f_1 \in RV_q$ and $q > \max\{1, p-1\}$, by Lemma A.5, we can choose $\Lambda > 1$ so large that

$$(\Lambda + \Lambda^{p-1})Cf_1(v + z) \leq \alpha_\varepsilon f_1(\Lambda(v + z)) = \alpha_\varepsilon f_1(\bar{\mathbf{u}}).$$

Since $\bar{\mathbf{u}} \geq z \geq c_1\phi(K(d(x)))$, by the first inequality of (3.4), $f_1(\bar{\mathbf{u}}) \leq k^p(d(x))f(\bar{\mathbf{u}})$. Hence

$$\bar{\mathbf{u}}_t - \Delta_p \bar{\mathbf{u}} \geq -\alpha_\varepsilon k^p(d(x))f(\bar{\mathbf{u}}) \geq -b(x, t)f(\bar{\mathbf{u}}), \quad (x, t) \in \Omega \times (0, T - \varepsilon].$$

Thus, we obtain (3.3) again.

Step 2: The existence of minimal solution.

Let $n \geq 1$ and consider the problem

$$\begin{cases} u_t - \Delta_p u = -b(x, t)f(u), & (x, t) \in \Omega_T, \\ u = n, & (x, t) \in \Sigma_T. \end{cases} \quad (3.9)$$

Since 0 and n are the lower and upper solutions of (3.9), it is clear that (3.9) has a unique positive solution $u_n(x, t)$ and $u_n(x, t)$ is non-increasing in t . Moreover, Proposition 2.1 guarantees that $u_n(x, t)$ is strictly increasing in n , that is, $u_n(x, t) < u_{n+1}(x, t)$ on Ω_T .

Let $\bar{\mathbf{u}}(x, t)$ be determined by Step 1. For any fixed n , it is clear that $u_n(x, t) < \bar{\mathbf{u}}(x, t)$ when (x, t) is near Σ_T . Since $\bar{\mathbf{u}}(x, t)$ satisfies (3.3), by Proposition 2.1 we have that $u_n(x, t) \leq \bar{\mathbf{u}}(x, t)$ in $\Omega_{T-\varepsilon}$. It should be noticed that, for fixed small $\varepsilon > 0$ and any compact subset Ω' of Ω , $\bar{\mathbf{u}}$ is bounded on $\Omega' \times [\varepsilon, T - \varepsilon]$. As a consequence, by standard regularity arguments, $u_n(x, t) \rightarrow \underline{u}(x, t)$ as $n \rightarrow \infty$ uniformly on any compact subset of $\Omega \times (0, T)$, where $\underline{u}(x, t)$ satisfies (1.1) in the weak sense. As $u_n(x, t)$ is non-increasing in t , so is $\underline{u}(x, t)$. Similar to the elliptic case, it can be easily proved that $\underline{u}(x, t) = \infty$ on Σ_T ; see e.g. [6]. Thus, $\underline{u}(x, t)$ is a solution to (1.1)–(1.3); in fact, it is the minimal positive solution. Indeed, let $u(x, t)$ be any positive solution of (1.1)–(1.3). We can easily apply Proposition 2.1 to conclude that $u_n(x, t) \leq u(x, t)$ in $\Omega \times (0, T]$. Letting $n \rightarrow \infty$ we deduce $\underline{u}(x, t) \leq u(x, t)$ in $\Omega \times (0, T)$.

Step 3: Existence of a maximal positive solution.

We next prove the existence of a maximal positive solution of (1.1)–(1.3). To achieve this, for any small $\varepsilon > 0$, we define $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$. Obviously, for small ε , $\partial\Omega_\varepsilon$ has the same smoothness as $\partial\Omega$. We consider the following problem:

$$\begin{cases} u_t - \Delta_p u = -b(x, t)f(u), & x \in \Omega_\varepsilon \times (\varepsilon, T), \\ u = \infty, & x \in \partial\Omega_\varepsilon \times (\varepsilon, T) \cup \bar{\Omega}_\varepsilon \times \{\varepsilon\}. \end{cases} \quad (3.10)$$

Let us denote by $\underline{u}^\varepsilon$ the minimal positive solution of (3.10). Proposition 2.1 guarantees that $\underline{u}^{\varepsilon_1} \geq \underline{u}^{\varepsilon_2} \geq \underline{u}$ in $\Omega_{\varepsilon_1} \times (\varepsilon_1, T)$ when $\varepsilon_1 > \varepsilon_2 > 0$. Therefore, one can construct a decreasing sequence ε_n satisfying $\varepsilon_n \rightarrow 0$, such that $\underline{u}^{\varepsilon_n} \rightarrow \bar{u}$ as $\varepsilon_n \rightarrow 0$ and \bar{u} solves (1.1)–(1.3). We further observe that \bar{u} is in fact the maximal positive solution. Indeed, for any positive solution u of (1.1)–(1.3), it follows from the comparison principle that $\underline{u}^{\varepsilon_n} > u$ in $\Omega_{\varepsilon_n} \times (\varepsilon_n, T)$ for each n . By taking $n \rightarrow \infty$ we obtain $\bar{u} \geq u$.

Step 4: Proof of (1.9).

Let $\xi(t)$ and $v(t)$ be the unique positive solution of (1.5) and (3.6) respectively. Since $f_2(s) \in RV_q$ and $q > 1$, by Lemma A.11, there is a constant $C > 0$ such that $C^{-1}v(t) \leq \xi^*(t) \leq Cv(t)$, here $\xi^*(t)$ is the unique positive solution of (1.8).

For any $0 < \delta \ll 1$, denote $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$. Let $z_\delta(x)$ be, respectively, the unique positive solution of

$$\begin{cases} \Delta_p z = k^p(d(x, \partial\Omega_\delta))f(z), & x \in \Omega_\delta, \\ z = \infty, & x \in \partial\Omega_\delta \end{cases}$$

when k is non-decreasing, and the unique positive solution of

$$\begin{cases} \Delta_p z = k^p(d(x))f(z), & x \in \Omega_\delta, \\ z = \infty, & x \in \partial\Omega_\delta \end{cases} \quad (3.11)$$

when k is non-increasing (see Theorem A.2(i), here we emphasize that for problem (3.11), the corresponding $k(t) = 1$). Set $\xi_\delta(t) = \xi(t - \delta)$ and $v_\delta(t) = v(t - \delta)$. From the discussion of Step 1, we can find a constant $\Lambda \geq 1$, which is independent of δ , such that the function

$$\mathbf{u}^\delta(x, t) = \begin{cases} \Lambda(\xi_\delta(t) + z_\delta(x)) & \text{if } k \text{ is non-increasing,} \\ \Lambda(v_\delta(t) + z_\delta(x)) & \text{if } k \text{ is non-decreasing} \end{cases}$$

satisfies

$$\mathbf{u}_t^\delta - \Delta_p \mathbf{u}^\delta \geq -b(x, t)f(\mathbf{u}^\delta), \quad (x, t) \in \Omega_\delta \times (\delta, T - \varepsilon].$$

It follows from the comparison principle that

$$\bar{u}(x, t) \leq \mathbf{u}^\delta(x, t) = \begin{cases} \Lambda(\xi_\delta(t) + z_\delta(x)) & \text{if } k \text{ is non-increasing,} \\ \Lambda(v_\delta(t) + z_\delta(x)) & \text{if } k \text{ is non-decreasing} \end{cases}$$

as $(x, t) \in \Omega_\delta \times (\delta, T - \varepsilon]$. Letting $\delta \rightarrow 0$, and using the easily proved fact that $z_\delta \rightarrow z$, $\xi_\delta \rightarrow \xi$ and $v_\delta \rightarrow v$, we deduce

$$\bar{u}(x, t) \leq \begin{cases} \Lambda(\xi(t) + z(x)) & \text{if } k \text{ is non-increasing,} \\ \Lambda(v(t) + z(x)) & \text{if } k \text{ is non-decreasing} \end{cases}$$

as $(x, t) \in \Omega \times (0, T - \varepsilon]$, where $z(x)$ is the unique positive solution of (3.1) and satisfies $z(x) \leq c_2\phi(K(d(x)))$. Thanks to $v(t) \leq C\xi^*(t)$, we conclude that (1.9) holds.

Step 5: Proof of (1.10).

Case (1): k is non-increasing.

Choose $\hat{\alpha}_\varepsilon > 0$ such that $\alpha_2(t) \leq \hat{\alpha}_\varepsilon$ on $[0, T - \varepsilon]$. Let $w(x)$ be the unique positive solution of

$$\begin{cases} \Delta_p w = \hat{\alpha}_\varepsilon k^p(d(x))f(w), & x \in \Omega, \\ w = \infty, & x \in \partial\Omega. \end{cases}$$

Then there exist positive constants d_1 and d_2 such that

$$d_1\phi(K(d(x))) \leq w(x) \leq d_2\phi(K(d(x))), \quad x \in \Omega. \quad (3.12)$$

Following the arguments of Step 1, we have that, for any $x \in \Omega$,

$$\begin{cases} k(d) \leq k \circ K^{-1} \circ \phi^{-1}(d_1^{-1}s) & \text{if } \phi(K(d)) \leq d_1^{-1}s, \\ k(d) \geq k \circ K^{-1} \circ \phi^{-1}(d_2^{-1}s) & \text{if } \phi(K(d)) \geq d_2^{-1}s, \end{cases}$$

and the functions

$$f_i^*(s) := (k \circ K^{-1} \circ \phi^{-1}(d_i^{-1}s))^p f(s)$$

satisfy $f_i^*(s) \in RV_q$ and

$$f_1^*(s) \geq k^p(d)f(s) \text{ if } s \geq d_1\phi(K(d)), \quad f_2^*(s) \leq k^p(d)f(s) \text{ if } s \leq d_2\phi(K(d)). \quad (3.13)$$

Let $\eta(t)$ be the unique positive solution of

$$\eta' = -f_2^*(\eta), \quad t > 0; \quad \eta(0) = \infty.$$

By Lemma A.11, there is a constant $C > 0$ such that

$$C^{-1}\eta(t) \leq \xi^*(t) \leq C\eta(t). \quad (3.14)$$

For $0 < \sigma \ll 1$, take $\tau = \min\{\eta(T + \sigma), d_1 \inf_\Omega \phi(K(d(x)))\} > 0$. Since $f_2^*(s) \in RV_q$ and $q > 1$, by Lemma A.8, there are a positive, continuous and increasing function $g \in RV_q$ and a constant $\sigma > 0$ such that $\sigma g(u) \leq f_2^*(u) \leq g(u)$ for all $u \geq \tau > 0$. By Lemma A.7, there is a constant $c > 0$, such that $g(u) + g(v) > cg(u + v)$ for all $u, v \geq \tau$. Similar to the discussion of Step 1, there is a constant $C > 0$ such that

$$C^{-1}f_2^*(u) \leq f_1^*(u) \leq Cf_2^*(u), \quad \forall u \geq \tau. \quad (3.15)$$

Let $\underline{u}(x, t) = \kappa(\eta^\sigma(t) + w(x))$, where $0 < \kappa \ll 1$ will be chosen later and $\eta^\sigma(t) = \eta(\sigma + t)$. Since $\eta^\sigma + w \geq w \geq d_1\phi(K(d)) \geq \tau$ and $g(u) \geq f_2^*(u)$ when $u \geq \tau$, by the first inequality of (3.13) and (3.15), there is a constant $\tilde{c} > 0$ such that $g(\eta^\sigma + w) \geq \tilde{c}k^p(d)f(\eta^\sigma + w)$ in $\Omega \times [0, T]$. Noting that $\eta^\sigma \geq \tau$ in $[0, T]$ and $\tau \leq d_1\phi(K(d)) \leq w \leq d_2\phi(K(d))$ in Ω , here $d = d(x)$, we have

$$\begin{aligned} \Delta_p \underline{u} - \underline{u}_t &= \kappa f_2^*(\eta^\sigma) + \kappa^{p-1} \hat{\alpha}_\varepsilon k^p(d)f(w) \\ &\geq \kappa f_2^*(\eta^\sigma) + \kappa^{p-1} \hat{\alpha}_\varepsilon f_2^*(w) \\ &\geq \sigma \min\{\kappa, \kappa^{p-1} \hat{\alpha}_\varepsilon\} [g(\eta^\sigma) + g(w)] \\ &\geq c\sigma \min\{\kappa, \kappa^{p-1} \hat{\alpha}_\varepsilon\} g(\eta^\sigma + w) \\ &\geq c\tilde{c}\sigma \min\{\kappa, \kappa^{p-1} \hat{\alpha}_\varepsilon\} k^p(d)f(\eta^\sigma + w) \\ &\geq c\tilde{c}\sigma \min\{\kappa, \kappa^{p-1} \hat{\alpha}_\varepsilon\} \kappa^{-l} k^p(d)f(\underline{u}) \text{ by condition (C)}. \end{aligned} \quad (3.16)$$

Since $l > \max\{1, p-1\}$, by (3.16), there is a $0 < \kappa \ll 1$ such that $c\tilde{c}\sigma \min\{\kappa, \kappa^{p-1}\hat{\alpha}_\varepsilon\}\kappa^{-l} \geq \hat{\alpha}_\varepsilon$. Consequently,

$$\underline{\mathbf{u}}_t - \Delta_p \underline{\mathbf{u}} \leq -\hat{\alpha}_\varepsilon k^p(d(x))f(\underline{\mathbf{u}}) \leq -b(x, t)f(\underline{\mathbf{u}}), \quad (x, t) \in \Omega \times (0, T - \varepsilon]. \quad (3.17)$$

For any given $n \geq 1$, by a standard argument (see [6]), the problem

$$\begin{cases} \Delta_p w = \hat{\alpha}_\varepsilon k^p(d(x))f(w), & x \in \Omega, \\ w = n, & x \in \partial\Omega \end{cases}$$

has a unique positive solution w_n , and $w_n \rightarrow w$ locally uniformly in Ω as $n \rightarrow \infty$. Since $f(s) > 0$ for $s > 0$, the maximum principle implies that $w_n \leq n$ on $\bar{\Omega}$. It follows that w_n is a lower solution of (3.9). Therefore, $u_n \geq w_n$ in Ω_T for all $n \geq 1$, and hence $\underline{\mathbf{u}} \geq w$ in Ω_T . We may assume that the constant κ , as determined above, satisfies $0 < \kappa < 1/2$. Then $\underline{\mathbf{u}} > 2\kappa w$ in Ω_T . Thus

$$\underline{\mathbf{u}} - \underline{\mathbf{u}} = \kappa(\eta^\sigma + w) - \underline{\mathbf{u}} < \kappa\eta^\sigma - \frac{1}{2}\underline{\mathbf{u}}, \quad (x, t) \in \Omega_T.$$

Therefore, $\limsup_{(x,t) \rightarrow \Sigma_T} [\underline{\mathbf{u}}(x, t) - \underline{\mathbf{u}}(x, t)] < 0$. Since $\underline{\mathbf{u}}$ satisfies (3.17), by the comparison principle, $\underline{\mathbf{u}} \geq \underline{\mathbf{u}} = \kappa(\eta^\sigma + w)$ in $\Omega_{T-\varepsilon}$. Taking $\sigma \rightarrow 0$ yields $\underline{\mathbf{u}} \geq \kappa(\eta + w)$ in $\Omega_{T-\varepsilon}$. Thanks to (3.12) and (3.14), and the arbitrariness of $\varepsilon > 0$, we obtain the first inequality of (1.10).

Case (2): k is non-decreasing.

For any small $\sigma > 0$, we consider the following auxiliary problems:

$$\xi' = -f(\xi), \quad t > -\sigma; \quad \xi(-\sigma) = \infty, \quad (3.18)$$

$$\begin{cases} \Delta_p z = k^p(d(x, \partial D_\sigma))f(z), & x \in D_\sigma, \\ z = \infty, & x \in \partial D_\sigma, \end{cases} \quad (3.19)$$

where $D_\sigma := \{x \in \mathbb{R}^N, d(x, \Omega) < \sigma\}$. We can choose σ sufficiently small such that ∂D_σ has the same smoothness as $\partial\Omega$. Denote by ξ^σ and z^σ the solutions of (3.18) and (3.19) respectively. It is easy to see that for any $t \in [0, T]$ and $x \in \Omega$, both ξ^σ and z^σ are decreasing in σ . Hence,

$$\tau = \min \left\{ \inf_{0 < \sigma \ll 1} \xi^\sigma(T), \inf_{x \in \Omega, 0 < \sigma \ll 1} z^\sigma(x) \right\} > 0.$$

As $f \in RV_\rho$ and f is increasing, there is a constant $c > 0$, such that $f(u) + f(v) > cf(u + v)$ for all $u, v \geq \tau > 0$. Since k is non-decreasing, we have $\tilde{c}k^p(d(x)) \leq 1$ for some $\tilde{c} > 0$. Set $\underline{\mathbf{u}}(x, t) = \kappa[\xi^\sigma(t) + z^\sigma(x)]$, where $\kappa > 0$ is to be determined. Noting that $d(x, \partial D_\sigma) > d(x, \partial\Omega)$ for all $x \in \Omega$, we have

$$\begin{aligned} \underline{\mathbf{u}}_t - \Delta_p \underline{\mathbf{u}} &= -\kappa f(\xi^\sigma) - \kappa^{p-1}k^p(d(x, \partial D_\sigma))f(z^\sigma) \\ &< -\min\{\tilde{c}\kappa, \kappa^{p-1}\}k^p(d(x))[f(\xi^\sigma) + f(z^\sigma)] \\ &\leq -c\min\{\tilde{c}\kappa, \kappa^{p-1}\}k^p(d(x))f(\xi^\sigma + z^\sigma), \quad (x, t) \in \Omega_{T-\varepsilon}. \end{aligned}$$

Similar to Case (1), there exists a suitably small $\kappa > 0$ such that

$$-c\min\{\tilde{c}\kappa, \kappa^{p-1}\}k^p(d(x))f(\xi^\sigma + z^\sigma) \leq -\hat{\alpha}_\varepsilon k^p(d(x))f(\underline{\mathbf{u}}) \leq -b(x, t)f(\underline{\mathbf{u}}), \quad (x, t) \in \Omega_{T-\varepsilon}.$$

By the comparison principle

$$\underline{u}(x, t) \geq \underline{\mathbf{u}}(x, t) = \kappa[\xi^\sigma(t) + z^\sigma(x)], \quad (x, t) \in \Omega_{T-\varepsilon}. \quad (3.20)$$

Clearly, $\xi^\sigma(t) \rightarrow \xi(t)$ locally uniformly on $(0, T]$ as $\sigma \rightarrow 0^+$ and $\xi(t)$ is the unique solution of (1.5). Similarly, $z^\sigma(x) \rightarrow z(x)$ locally uniformly on any compact subset of Ω as $\sigma \rightarrow 0^+$, and $z(x)$ is the unique positive solution of (3.1). Letting $\sigma \rightarrow 0^+$ in (3.20), and using (3.2), the desired result is obtained since $\varepsilon > 0$ is arbitrary. \square

4 Asymptotic behavior and uniqueness

In this section, we prove Theorems 1.2 and 1.3. We first prove a lemma. Since $\rho > p - 1 - (p - 2)/\ell$, it is easy to check that $p(1 - \ell)/(r - 1) < \rho - 1$.

Lemma 4.1 *For any given constant $\varsigma > 0$ where $p(1 - \ell)/(r - 1) < \varsigma < \rho - 1$, we have*

$$\lim_{s \rightarrow 0^+} \frac{\phi^{-\varsigma}(K(s))}{k^p(s)} = 0. \quad (4.1)$$

Proof We recall that $\phi \in NRVZ_{1-r}$ (Lemma A.2), $K \in RVZ_{1/\ell}$ and $k \in RVZ_{(1-\ell)/\ell}$ (Lemma A.1). In view of Lemma A.4,

$$\phi^{-\varsigma}(K(s)) \in RVZ_{\varsigma(r-1)/\ell}, \quad k^p(s) \in RVZ_{p(1-\ell)/\ell}, \quad \frac{\phi^{-\varsigma}(K(s))}{k^p(s)} \in RVZ_{\varsigma(r-1)/\ell - p(1-\ell)/\ell}.$$

Since $\varsigma > p(1 - \ell)/(r - 1)$, i.e., $\varsigma(r - 1)/\ell - p(1 - \ell)/\ell > 0$, it is obvious that (4.1) holds. \square

Proof of Theorem 1.2(i) Fix $y \in \partial\Omega$ and $t_0 \in (0, T)$, and let $\beta_0 := \beta(y, t_0)$. For any given small $\varepsilon \in (0, \beta_0/2)$, one can find a sufficiently small constant $\delta \in (0, t_0)$ such that, for $(x, t) \in \Omega_T$ satisfying $|x - y| < \delta$ and $|t - t_0| < \delta$, we have

$$\beta_0 - \varepsilon \leq \frac{b(x, t)}{k^p(d(x))} \leq \beta_0 + \varepsilon.$$

Step 1: We first prove the upper bound estimate

$$\limsup_{\Omega \ni x \rightarrow y} \frac{u(x, t_0)}{\phi(K(d(x)))} \leq \left(\frac{r + \ell - 1}{r(\beta_0 - 2\varepsilon)} \right)^{\frac{r-1}{p}}. \quad (4.2)$$

Let $\eta(t)$ be the unique positive solution of

$$\begin{cases} \eta'(t) = -af(\eta), & t \in (t_0 - \delta, t_0], \\ \eta(t_0 - \delta) = \infty, & \eta(t_0) = 1, \end{cases}$$

where $a = \frac{1}{\delta} \int_1^\infty \frac{ds}{f(s)} > 0$. Then $\eta(t) \geq 1$ on $(t_0 - \delta, t_0]$. Let ς be given by Lemma 4.1. Since f is increasing, $f \in RV_\rho$ and $\rho > \varsigma + 1$, by Lemma A.6, there is a constant $\Lambda^* > 0$ such that

$$a\Lambda^{\varsigma+1}f(\eta(t)) < \varepsilon f(\Lambda\eta(t)), \quad \forall \Lambda \geq \Lambda^*, \quad t \in (t_0 - \delta, t_0]. \quad (4.3)$$

Let $w_\varepsilon(x)$ be the unique positive solution of

$$\begin{cases} \Delta_p w_\varepsilon = (\beta_0 - 2\varepsilon)k^p(d(x))f(w_\varepsilon), & x \in \Omega, \\ w_\varepsilon = \infty, & x \in \partial\Omega. \end{cases}$$

By Theorem A.2, there are two positive constants d_1 and d_2 such that

$$d_1\phi(K(d(x))) \leq w_\varepsilon(x) \leq d_2\phi(K(d(x))), \quad x \in \Omega. \quad (4.4)$$

Therefore $\lim_{\gamma \rightarrow 0^+} \inf_{\Omega \cap B_{2\gamma}(y)} w_\varepsilon(x) = \infty$. There is a constant γ_0 with $0 < \gamma_0 \leq \delta$ such that

$$w_\varepsilon(x) > \Lambda^*, \quad \forall x \in \Omega \cap B_{2\gamma}(y), \quad 0 < \gamma \leq \gamma_0. \quad (4.5)$$

For any fixed $0 < \gamma \leq \gamma_0$, let $D \subset \Omega \cap B_{2\gamma}(y)$ be a smooth domain such that ∂D and $\partial\Omega$ coincide inside $B_\gamma(y)$. Let $v_\varepsilon(x)$ be a positive solution of

$$\begin{cases} \Delta_p v_\varepsilon = (\beta_0 - 2\varepsilon)k^p(d(x))f(v_\varepsilon), & x \in D, \\ v_\varepsilon = \infty, & x \in \partial D. \end{cases}$$

We note that since $d(x) = d(x, \partial\Omega)$, which may not be equal to $d(x, \partial D)$, the positive solution of the above problem may not be unique. As $D \subset \Omega \cap B_{2\gamma}(y)$, by the comparison principle we have $v_\varepsilon(x) \geq w_\varepsilon(x)$ in D . Hence, by (4.5),

$$v_\varepsilon(x) \geq w_\varepsilon(x) > \Lambda^*, \quad \forall x \in D. \quad (4.6)$$

From the choice of D , it is clear that $d(x) = d(x, \partial D)$ when $x \in D$ and is near y . Evidently,

$$\lim_{D \ni x \rightarrow y} \frac{(\beta_0 - 2\varepsilon)k^p(d(x))}{k^p(d(x, \partial D))} = \beta_0 - 2\varepsilon.$$

In view of Remark A.1 we have

$$\lim_{D \ni x \rightarrow y} \frac{v_\varepsilon(x)}{\phi(K(d(x)))} = \lim_{D \ni x \rightarrow y} \frac{v_\varepsilon(x)}{\phi(K(d(x, \partial D)))} = \left(\frac{r + \ell - 1}{r(\beta_0 - 2\varepsilon)} \right)^{\frac{r-1}{p}}. \quad (4.7)$$

We now consider $\Omega_\sigma := \{x \in \Omega : d(x) \geq \sigma\}$ for sufficiently small $\sigma \in [0, \gamma/2)$. For each such Ω_σ , we can construct a smooth domain $D_\sigma \subset \Omega_\sigma \cap B_{2\gamma}(y) \subset D$ such that ∂D_σ and $\partial\Omega_\sigma$ coincide inside $B_\gamma(y)$, and D_σ varies continuously with σ for all small non-negative σ . We may also require that $D_\sigma \subset D_{\sigma'}$ when $\sigma > \sigma'$ and $D_\sigma \rightarrow D$ as $\sigma \rightarrow 0^+$. By Theorem A.2, the problem

$$\begin{cases} \Delta_p v_\sigma = (\beta_0 - 2\varepsilon)k^p(d(x))f(v_\sigma), & x \in D_\sigma, \\ v_\sigma = \infty, & x \in \partial D_\sigma \end{cases}$$

has a unique positive solution, denoted by v_σ (with $k(t) = 1$, $\beta(y) = (\beta_0 - 2\varepsilon)k^p(d(y))$ for $y \in \partial D_\sigma$). Applying the comparison principle and (4.6) we get $v_\sigma(x) \geq v_\varepsilon(x) \geq w_\varepsilon(x) > \Lambda^*$ in D_σ . By further using the elliptic regularity, we see that v_σ decreases to v_ε as σ decreases to 0.

Set $u_\sigma(x, t) = \eta(t)v_\sigma(x)$. Then for $(x, t) \in D_\sigma \times (t_0 - \delta, t_0]$,

$$\begin{aligned} (u_\sigma)_t - \Delta_p u_\sigma &= \eta' v_\sigma - \eta^{p-1} \Delta_p v_\sigma \\ &= -av_\sigma f(\eta) - (\beta_0 - 2\varepsilon)k^p(d(x))\eta^{p-1}(t)f(v_\sigma) \\ &= -av_\sigma^{-\varsigma} v_\sigma^{\varsigma+1} f(\eta) - (\beta_0 - 2\varepsilon)k^p(d(x))\eta^{p-1} f(v_\sigma). \end{aligned} \quad (4.8)$$

Thanks to the facts that $f(s)/s^{p-1}$ is increasing and $\eta \geq 1$, one has

$$\eta^{p-1}f(v_\sigma) \leq f(\eta v_\sigma) = f(u_\sigma), \quad (x, t) \in D_\sigma \times (t_0 - \delta, t_0]. \quad (4.9)$$

Noting that $v_\sigma(x) > \Lambda^*$ and $\varsigma > 0$, and taking into account (4.3), we have

$$av_\sigma^{\varsigma+1}f(\eta) < \varepsilon f(v_\sigma\eta) = \varepsilon f(u_\sigma), \quad (x, t) \in D_\sigma \times (t_0 - \delta, t_0]. \quad (4.10)$$

If $k(0) > 0$, then $k^p(d(x))$ has a positive lower bounded in Ω . In view of $\lim_{\gamma \rightarrow 0^+} \inf_{\Omega \cap B_{2\gamma}(y)} w_\varepsilon = \infty$, we can choose γ small enough such that $w_\varepsilon^{-\varsigma}(x) < k^p(d(x))$. Hence

$$v_\sigma^{-\varsigma}(x) < k^p(d(x)), \quad \forall x \in D_\sigma. \quad (4.11)$$

If $k(0) = 0$, by Lemma 4.1,

$$\lim_{d(x) \rightarrow 0^+} \frac{\phi^{-\varsigma}(K(d(x)))}{k^p(d(x))} = 0. \quad (4.12)$$

In view of $v_\sigma(x) \geq w_\varepsilon(x)$ in D_σ and the estimates (4.4), it follows that $v_\sigma(x) \geq w_\varepsilon(x) \geq d_1\phi(K(d(x)))$ in D_σ . Therefore

$$\frac{v_\sigma^{-\varsigma}(x)}{k^p(d(x))} \leq d_1^{-\varsigma} \frac{\phi^{-\varsigma}(K(d(x)))}{k^p(d(x))}, \quad \forall x \in D_\sigma. \quad (4.13)$$

It is clear that $d(x) \rightarrow 0^+$ holds uniformly on \overline{D}_σ as $\gamma \rightarrow 0^+$. By virtue of (4.12) and (4.13), one can choose γ small enough such that (4.11) is true.

It follows from (4.8)–(4.12) that ($d = d(x)$)

$$\begin{aligned} (u_\sigma)_t - \Delta_p u_\sigma &= -av_\sigma^{-\varsigma}v_\sigma^{\varsigma+1}f(\eta) - (\beta_0 - 2\varepsilon)k^p(d)\eta^{p-1}f(v_\sigma) \\ &\geq -(\beta_0 - \varepsilon)k^p(d)f(u_\sigma) \\ &\geq -bf(u_\sigma), \quad (x, t) \in D_\sigma \times (t_0 - \delta, t_0]. \end{aligned} \quad (4.14)$$

It is obvious that

$$\begin{aligned} u(x, t_0 - \delta) &< u_\sigma(x, t_0 - \delta), \quad x \in D_\sigma, \\ u(x, t)|_{\partial D_\sigma} &< u_\sigma(x, t)|_{\partial D_\sigma}, \quad t \in (t_0 - \delta, t_0]. \end{aligned}$$

By (4.14) and the comparison principle, $u(x, t) \leq u_\sigma(x, t) = \eta(t)v_\sigma(x)$ in $D_\sigma \times (t_0 - \delta, t_0]$. Letting $\sigma \rightarrow 0$, one has $u(x, t) \leq \eta(t)v_\varepsilon(x)$ in $D \times (t_0 - \delta, t_0]$. Hence, by (4.7),

$$\limsup_{\Omega \ni x \rightarrow y} \frac{u(x, t_0)}{\phi(K(d(x)))} \leq \limsup_{D \ni x \rightarrow y} \frac{v_\varepsilon(x)}{\phi(K(d(x)))} = \left(\frac{r + \ell - 1}{r(\beta_0 - 2\varepsilon)} \right)^{\frac{r-1}{p}}.$$

We thus obtain the estimate (4.2).

Step 2: Now we establish the lower bound estimate

$$\liminf_{\Omega \ni x \rightarrow y} \frac{u(x, t_0)}{\phi(K(d(x)))} \geq \left(\frac{r + \ell - 1}{r(\beta_0 + \varepsilon)} \right)^{\frac{r-1}{p}}. \quad (4.15)$$

Choose a constant $A_0 > 0$ such that $\alpha_2(t) \leq A_0$ in $[0, t_0]$. Let w and z be the unique positive solutions of following problems, respectively:

$$\begin{aligned}\Delta_p w &= A_0 k^p(d(x))f(w), \quad x \in \Omega; \quad w = \infty, \quad x \in \partial\Omega, \\ \Delta_p z &= (\beta_0 + \varepsilon)k^p(d(x))f(z), \quad x \in \Omega; \quad z = \infty, \quad x \in \partial\Omega.\end{aligned}$$

According to Theorem A.2, there is a constant $C > 0$ such that

$$C^{-1}w(x) \leq z(x) \leq Cw(x), \quad \forall x \in \Omega. \quad (4.16)$$

Let w_n be the unique positive solution of

$$\begin{cases} \Delta_p w = A_0 k^p(d(x))f(w), & x \in \Omega, \\ w = n, & x \in \partial\Omega. \end{cases}$$

Then w_n is increasing in n and $w_n \rightarrow w$ uniformly on any compact subset of Ω . Thanks to $b(x, t) \leq \alpha_2(t)k^p(d(x)) \leq A_0 k^p(d(x))$ for all $(x, t) \in \Omega \times [0, t_0]$, we see that w_n satisfies

$$\begin{cases} \Delta_p w_n = A_0 k^p(d(x))f(w_n) \geq b(x, t)f(w_n), & x \in \Omega, \\ w_n = n, & x \in \partial\Omega. \end{cases}$$

Proposition 2.1 asserts that $w_n \leq u$ in $\Omega \times [0, t_0]$ for all n . Hence, $w \leq u$ in $\Omega \times [0, t_0]$. This combines with (4.16) to yield $z \leq Cw \leq Cu$ in $\Omega \times [0, t_0]$. Denote $\zeta_0 = C^{-1}$. Then we have

$$\zeta_0 z(x) \leq u(x, t_0 - \delta), \quad x \in \Omega. \quad (4.17)$$

Let ζ be the unique positive solution of

$$\begin{cases} \zeta'(t) = \tilde{a}f(\zeta), & t \in (t_0 - \delta, t_0], \\ \zeta(t_0 - \delta) = \zeta_0, \quad \zeta(t_0) = 1, \end{cases}$$

where $\tilde{a} = \frac{1}{\delta} \int_{\zeta_0}^1 \frac{ds}{f(s)} > 0$. Then $\zeta(t) \leq 1$ on $[t_0 - \delta, t_0]$.

We first consider the case $1 < p < 2$. As above, let $D \subset \Omega \cap B_{2\delta}(y)$ be a smooth domain such that ∂D and $\partial\Omega$ coincide inside $B_\delta(y)$. Take $\psi = \frac{1}{2}z|_{\partial D}$, and let $\{\psi_n\}_{n=1}^\infty$ be an increasing sequence of non-negative smooth functions defined on ∂D with the property that

$$\psi_n|_{\overline{\partial D \cap \partial\Omega}} = n \quad \text{and} \quad \psi_n \rightarrow \psi \text{ uniformly on any compact subset of } \partial D \setminus \overline{\partial D \cap \partial\Omega}.$$

Let $A > \tilde{a}f(1)$ be a given constant. Then for any $m \geq 1$, the problem

$$\Delta_p v = Av + (\beta_0 + \varepsilon) \min\{m, k^p(d(x))\}f(v), \quad x \in D; \quad v|_{\partial D} = \psi_n$$

has a the unique positive solution, denoted by v_n^m ; cf. [6]. By the comparison principle, $v_n^m \geq v_n^{m+1}$. Thus $v_n = \lim_{m \rightarrow \infty} v_n^m$ exists, and one easily sees by standard elliptic regularity that v_n is a solution to

$$\begin{cases} \Delta_p v = Av + (\beta_0 + \varepsilon)k^p(d(x))f(v), & x \in D, \\ v = \psi_n, & x \in \partial D. \end{cases}$$

The comparison principle infers that v_n is unique, $v_n \leq v_{n+1}$ in D since $\psi_n \leq \psi_{n+1}$ on ∂D , and $v_n \leq v^*$, where v^* is the unique positive solution of

$$\begin{cases} \Delta_p v = Av + (\beta_0 + \varepsilon)k^p(d(x))f(v), & x \in D, \\ v = \infty, & x \in \partial D. \end{cases} \quad (4.18)$$

Since $p < 2$ and $A > 0$, the function $As/s^{p-1} = As^{2-p}$ is increasing in $s > 0$, and hence the comparison principle holds for the problem (4.18). The existence and uniqueness of v^* can be proved by the similar methods of [6, 15].

Thus $v := \lim_{n \rightarrow \infty} v_n$ exists, and by the elliptic regularity we find that v is a positive solution of

$$\begin{cases} \Delta_p v = Av + (\beta_0 + \varepsilon)k^p(d(x))f(v), & x \in D, \\ v = \psi = \frac{1}{2}z, & x \in \partial D. \end{cases} \quad (4.19)$$

In fact, by the interior regularity it is easy to show that v satisfies the differential equation of (4.19). Using the boundary estimate we can prove that v is continuous in $\overline{D} \setminus \overline{\partial D \cap \partial \Omega}$. Hence, $v = \psi$ in $\partial D \setminus \overline{\partial D \cap \partial \Omega}$ in the classical sense. Now we prove that for any $x_0 \in \overline{\partial D \cap \partial \Omega}$, the limit $\lim_{D \ni x \rightarrow x_0} v(x) = \infty$ holds. If this is not true, then there exist $x_0 \in \overline{\partial D \cap \partial \Omega}$, a sequence $\{x_l\}_{l=1}^\infty \subset D$ and a constant $M > 0$, such that $x_l \rightarrow x_0$ and $v(x_l) \leq M$. Since $v_n \leq v$ for all n , we have $v_n(x_l) \leq M$ for all n and l . Letting $l \rightarrow \infty$, we see that $v_n(x_0) \leq M$ for all n . This is a contradiction since $v_n(x_0) = \psi_n(x_0) = n$ for all n .

The comparison principle asserts $v \leq z$ in D . Because the comparison principle holds for the problem (4.19), similar to the proof of [6, Theorem 1.2(i)] or [15, Theorem 1.2], we also have that

$$\liminf_{D \ni x \rightarrow y} \frac{v(x)}{\phi(K(d(x)))} = \left(\frac{r + \ell - 1}{r(\beta_0 + \varepsilon)} \right)^{\frac{r-1}{p}}. \quad (4.20)$$

Set $u^*(x, t) = \zeta(t)v(x)$ for $(x, t) \in D \times [t_0 - \delta, t_0]$. Clearly, by (4.17), $u^*(x, t_0 - \delta) = \zeta_0 v(x) \leq \zeta_0 z(x) \leq u(x, t_0 - \delta)$ in D . It is also evident that $u^* \leq u$ on $\partial D \times [t_0 - \delta, t_0]$. A direct computation yields

$$\begin{aligned} u_t^* - \Delta_p u^* &= \zeta' v - \zeta^{p-1} \Delta_p v \\ &= \tilde{a} f(\zeta) v - \zeta^{p-1} [Av + (\beta_0 + \varepsilon)k^p(d(x))f(v)] \\ &= (\tilde{a} f(\zeta) - A\zeta^{p-1}) v - (\beta_0 + \varepsilon)k^p(d(x))\zeta^{p-1} f(v). \end{aligned}$$

Thanks to the facts that $f(s)/s^{p-1}$ is increasing in $s > 0$ and $\zeta(t) \leq 1$ in $[t_0 - \delta, t_0]$, one has $f(\zeta) \leq f(1)\zeta^{p-1}$ and $\zeta^{p-1} f(v) \geq f(\zeta v) = f(u^*)$. As $A > \tilde{a}f(1)$, it follows that

$$u_t^* - \Delta_p u^* \leq -(\beta_0 + \varepsilon)k^p(d(x))f(u^*), \quad (x, t) \in D \times [t_0 - \delta, t_0].$$

We can apply the comparison principle to conclude that $u^* \leq u$ in $D \times [t_0 - \delta, t_0]$. In particular, $v(x) = u^*(x, t_0) \leq u(x, t_0)$ in D . By (4.20), it follows that

$$\liminf_{\Omega \ni x \rightarrow y} \frac{u(x, t_0)}{\phi(K(d(x)))} \geq \liminf_{D \ni x \rightarrow y} \frac{v(x)}{\phi(K(d(x)))} = \left(\frac{r + \ell - 1}{r(\beta_0 + \varepsilon)} \right)^{\frac{r-1}{p}}.$$

Hence (4.15) holds. The desired result clearly follows from (4.2) and (4.15), since $\varepsilon > 0$ can be arbitrarily small.

Next, we consider the case $p \geq 2$. As above, let $A > \tilde{a}f(1)$ be a given constant. By arguments similar to those of [11, Theorem 4.4] and [6, Theorem 1.2], it can be proved that the problem

$$\begin{cases} \Delta_p z = Az^{p-1} + (\beta_0 + \varepsilon)k^p(d(x))f(z), & x \in \Omega, \\ z = \infty, & x \in \partial\Omega \end{cases}$$

has a unique positive solution, denoted by \hat{z} . The comparison principle yields $\hat{z} \leq z$ in Ω , and hence

$$\zeta_0 \hat{z}(x) \leq \zeta_0 z(x) \leq u(x, t_0 - \delta), \quad x \in \Omega \quad (4.21)$$

by (4.17). Moreover, there are two positive constants d'_1 and d'_2 such that

$$d'_1 \phi(K(d(x))) \leq \hat{z}(x) \leq d'_2 \phi(K(d(x))), \quad x \in \Omega.$$

Therefore $\lim_{\gamma \rightarrow 0^+} \inf_{\Omega \cap B_{2\gamma}(y)} \hat{z}(x) = \infty$. There is a constant $\gamma > 0$ such that

$$\hat{z}(x) > 2, \quad \forall x \in \Omega \cap B_{2\gamma}(y). \quad (4.22)$$

As above, let $\hat{D} \subset \Omega \cap B_{2\gamma}(y)$ be a smooth domain such that $\partial\hat{D}$ and $\partial\Omega$ coincide inside $B_\gamma(y)$. Similar to the above, the problem

$$\begin{cases} \Delta_p v = Av^{p-1} + (\beta_0 + \varepsilon)k^p(d(x))f(v), & x \in \hat{D}, \\ v = \frac{1}{2}\hat{z}, & x \in \partial\hat{D} \end{cases} \quad (4.23)$$

has a positive solution, denoted by \hat{v} , and \hat{v} satisfies (4.20). Moreover, by the comparison principle, $\hat{v} \leq \hat{z}$ in \hat{D} . Since the function $w = \frac{1}{2}\hat{z}$ satisfies

$$\Delta_p w \geq Aw^{p-1} + (\beta_0 + \varepsilon)k^p(d(x))f(w),$$

the comparison principle gives $\hat{v} \geq w = \frac{1}{2}\hat{z}$ in \hat{D} . Hence $\hat{v} > 1$ in \hat{D} by (4.22).

Set $\hat{u}(x, t) = \zeta(t)\hat{v}(x)$ for $(x, t) \in \hat{D} \times [t_0 - \delta, t_0]$. Then, as $\hat{v} > 1$ in \hat{D} and $p \geq 2$, similar to the above, we have that

$$\begin{aligned} \hat{u}_t - \Delta_p \hat{u} &= \tilde{a}f(\zeta)\hat{v} - \zeta^{p-1}[A\hat{v}^{p-1} + (\beta_0 + \varepsilon)k^p(d(x))f(\hat{v})] \\ &= (\tilde{a}f(\zeta) - A\zeta^{p-1}\hat{v}^{p-2})\hat{v} - (\beta_0 + \varepsilon)k^p(d(x))\zeta^{p-1}f(\hat{v}) \\ &\leq -(\beta_0 + \varepsilon)k^p(d(x))f(\hat{u}), \quad (x, t) \in \hat{D} \times [t_0 - \delta, t_0]. \end{aligned}$$

Since $\hat{v}(x)$ satisfies (4.20), the rest of the proof is the same as that of the case $1 < p < 2$. \square

Proof of Theorem 1.2(ii) Let $x_0 \in \Omega$ be fixed. Then, for any given small $\varepsilon > 0$, we can find a small ball $B_r(x_0)$ and small $t_0 > 0$ such that $\bar{B}_r(x_0) \subset \Omega$ and

$$0 < b_0 - \varepsilon \leq b(x, t) \leq b_0 + \varepsilon$$

for all $x \in B_r(x_0)$, $t \in [0, t_0]$, where $b_0 = b(x_0, 0)$.

Step 1 Let $\mu_\varepsilon^*(t)$ be the unique positive solution of

$$(\mu_\varepsilon^*)' = -(b_0 - \varepsilon)f(\mu_\varepsilon^*), \quad t > 0; \quad \mu_\varepsilon^*(0) = \infty.$$

We shall prove that

$$\limsup_{t \rightarrow 0^+} \frac{u(x_0, t)}{\mu_\varepsilon^*(t)} \leq 1. \quad (4.24)$$

By Proposition A.1, $u^{-\rho} f(u) = M(u) \exp \left\{ \int_b^u \frac{\varphi(t)}{t} dt \right\}$, where $M(u)$ satisfies $\lim_{u \rightarrow \infty} M(u) = M^* > 0$. It follows that there is a sequence $\{a_n\}$ with $\lim_{n \rightarrow \infty} a_n = \infty$, such that $\frac{n-1}{n} M^* \leq M(u) \leq \frac{n}{n-1} M^*$ when $u \geq a_n$. Define

$$f_0(u) = M^* u^\rho \exp \left\{ \int_b^u \frac{\varphi(t)}{t} dt \right\}.$$

Then $f_0(u)$ satisfies

$$\frac{n-1}{n} f_0(u) \leq f(u) \leq \frac{n}{n-1} f_0(u), \quad \forall u \geq a_n. \quad (4.25)$$

Let $\nu > 0$ be small such that $1 + \nu < \rho$. Note that $\rho > p - 1$ and $\rho > 1 + \nu$, similar to the proof of Lemma A.8, we can prove that $f_0(u)/u^{1+\nu}$ is increasing when $u \geq A \gg 1$. Certainly, $f_0(u)/u$ is increasing for $u \geq A$, and hence

$$f_0(u) + f_0(v) \leq f_0(u + v), \quad \forall u, v \geq A. \quad (4.26)$$

It can be assumed that $a_n \geq A$ for all n without loss of generality.

Let $\eta_n(t)$, $\zeta_n(t)$, $z_n(x)$ and $\hat{z}(x)$ be solutions of the following problems, respectively:

$$\begin{aligned} \eta'_n &= -(b_0 - \varepsilon) \frac{n-1}{n} f_0(\eta_n), \quad t > 0; \quad \eta_n(0) = \infty; \\ \zeta'_n &= -(b_0 - \varepsilon) \frac{n}{n-1} f_0(\zeta_n), \quad t > 0; \quad \zeta_n(0) = \infty; \\ \Delta_p z_n &= (b_0 - \varepsilon) \frac{n-1}{n} f_0(z_n), \quad x \in B_r(x_0); \quad z_n = \infty, \quad x \in \partial B_r(x_0); \\ \Delta_p \hat{z} &= (b_0 - \varepsilon) f_0(\hat{z}), \quad x \in B_r(x_0); \quad \hat{z} = \infty \quad x \in \partial B_r(x_0). \end{aligned}$$

By a simple comparison argument, we have $z_n(x) \geq \hat{z}(x)$ in $B_r(x_0)$. Similar to the proof of Theorem 6.1 in [10], we can prove that $\lim_{r \rightarrow 0^+} \min_{\overline{B_r(x_0)}} \hat{z}(x) = \infty$. Choosing r small enough, we may assume that $\hat{z}(x) > A$, and hence $z_n(x) > A$ on $\overline{B_r(x_0)}$. It is also evident that there is $t_n > 0$ such that $\eta_n(t) \geq a_n$, $\zeta_n(t) \geq a_n$ for all $0 < t \leq t_n$.

Define $u_n(x, t) = \eta_n(t) + z_n(x)$. By (4.26) we have that, for $(x, t) \in B_r(x_0) \times (0, t_n]$,

$$\begin{aligned} (u_n)_t - \Delta_p u_n &= -(b_0 - \varepsilon) \frac{n-1}{n} [f_0(\eta_n) + f_0(z_n)] \\ &\geq -(b_0 - \varepsilon) \frac{n-1}{n} f_0(\eta_n + z_n) \\ &\geq -(b_0 - \varepsilon) f(\eta_n + z_n) \\ &= -(b_0 - \varepsilon) f(u_n). \end{aligned}$$

By the comparison principle we obtain

$$u(x, t) \leq u_n(x, t) = \eta_n(t) + z_n(x), \quad (x, t) \in B_r(x_0) \times (0, t_n]. \quad (4.27)$$

Thanks to (4.25), by a simple comparison argument, we have $\zeta_n(t) \leq \mu_\varepsilon^*(t) \leq \eta_n(t)$ in $(0, t_n]$. Let $l_n = [n/(n-1)]^{2/\nu} > 1$. If we can prove $\eta_n(t) \leq l_n \zeta_n(t)$ in $(0, t_n]$, then

$$\eta_n(t) \leq l_n \zeta_n(t) \leq l_n \mu_\varepsilon^*(t), \quad t \in (0, t_n]. \quad (4.28)$$

Since $f_0(u)/u^{1+\nu}$ is increasing for $u \geq A$ and $\zeta_n(t) \geq a_n \geq A$ for $0 < t \leq t_n$, it follows that $l_n^{1+\nu} f_0(\zeta_n(t)) \leq f_0(l_n \zeta_n(t))$ for all $0 < t \leq t_n$. Therefore

$$(l_n \zeta_n)' = -l_n(b_0 - \varepsilon) \frac{n}{n-1} f_0(\zeta_n) \geq -(b_0 - \varepsilon) \frac{n-1}{n} f_0(l_n \zeta_n) = \eta_n', \quad 0 < t \leq t_n.$$

Consequently, $\eta_n(t) \leq l_n \zeta_n(t)$ in $(0, t_n]$ by the comparison principle. By (4.27) and (4.28), $u(x, t) \leq l_n \mu_\varepsilon^*(t) + z_n(x)$ in $B_r(x_0) \times (0, t_n]$. Hence, $\limsup_{t \rightarrow 0^+} \frac{u(x_0, t)}{\mu_\varepsilon^*(t)} \leq l_n$ for all n . Taking $n \rightarrow \infty$, we obtain (4.24).

Choose $0 < \varepsilon_n \rightarrow 0^+$ and set $\varepsilon = \varepsilon_n$. Taking into account that $\lim_{t \rightarrow 0^+} \mu_{\varepsilon_n}^*(t) = \infty$, similar to the above arguments we can prove that there are $\delta_n \rightarrow 0^+$ and $t'_n \rightarrow 0^+$ such that

$$\mu_{\varepsilon_n}^*(t) \leq (1 + \delta_n) \tau(t), \quad \forall t \in (0, t'_n), \quad n \gg 1, \quad (4.29)$$

where $\tau(t)$ is the unique positive solution of (1.11). It is deduced from (4.24) and (4.29) that $\limsup_{t \rightarrow 0^+} \frac{u(x_0, t)}{\tau(t)} \leq 1 + \delta_n$ for all n . The limit (1.12) is obtained by take $n \rightarrow \infty$.

Step 2 Let $\tilde{\mu}_\varepsilon(t)$ be the positive solution of

$$\tilde{\mu}_\varepsilon' = -(b_0 + \varepsilon) f(\tilde{\mu}_\varepsilon), \quad t > 0; \quad \tilde{\mu}_\varepsilon(0) = \infty.$$

Under the conditions that $p > 2N/(N+2)$ and $f(s)/s$ is increasing for $s > 0$, we shall prove that

$$\liminf_{t \rightarrow 0^+} \frac{u(x_0, t)}{\tilde{\mu}_\varepsilon(t)} \geq 1. \quad (4.30)$$

We first consider the case $p \geq 2$. Let λ_1 be the first eigenvalue of the problem

$$\begin{cases} -\Delta_p \varphi = \lambda_1 \varphi^{p-1}, & x \in B_r(x_0), \\ \varphi = 0, & x \in \partial B_r(x_0), \end{cases}$$

and φ with $\sup_{B_r(x_0)} \varphi(x) = 1$ be the positive eigenfunction corresponding to λ_1 . Then $\lambda_1 > 0$. Obviously, $0 < \varphi(x) < 1$ in $B_r(x_0) \setminus \{x_0\}$ and $\varphi(x_0) = 1$. Let μ_ε be the unique positive solution of

$$\mu_\varepsilon' = -\lambda_1 \mu_\varepsilon^{p-1} - (b_0 + \varepsilon) f(\mu_\varepsilon), \quad t > 0; \quad \mu_\varepsilon(0) = \infty.$$

For any $\sigma : 0 < \sigma \ll 1$, set $\omega(x, t) = \mu_\varepsilon(\sigma + t) \varphi(x)$. Since $0 \leq \varphi(x) \leq 1$ and $f(s)/s$ is increasing in $s > 0$, it follows that

$$\begin{aligned} \omega_t - \Delta_p \omega &= \varphi \mu_\varepsilon'(\sigma + t) - \mu_\varepsilon^{p-1}(\sigma + t) \Delta_p \varphi \\ &= \lambda_1 \mu_\varepsilon^{p-1}(\sigma + t) (\varphi^{p-1} - \varphi) - (b_0 + \varepsilon) \varphi f(\mu_\varepsilon) \\ &\leq -(b_0 + \varepsilon) f(\mu_\varepsilon \varphi) = -(b_0 + \varepsilon) f(\omega), \quad (x, t) \in B_r(x_0) \times [0, t_0]. \end{aligned}$$

Noting that $\varphi = 0$ on $\partial B_r(x_0)$, and $u = \infty$ on $\overline{B_r(x_0)} \times \{0\}$, the above inequality shows that $\omega(x, t)$ is a lower solution of the problem

$$\begin{cases} v_t - \Delta_p v = -b(x, t) f(v), & (x, t) \in B_r(x_0) \times (0, t_0), \\ v = u, & (x, t) \in \partial B_r(x_0) \times (0, t_0) \cup \overline{B_r(x_0)} \times \{0\}. \end{cases} \quad (4.31)$$

Clearly u solves (4.31). The comparison argument then implies $\mu_\varepsilon(\sigma + t)\varphi(x) \leq u(x, t)$ in $B_r(x_0) \times (0, t_0]$. Letting $\sigma \rightarrow 0^+$, we deduce $\mu_\varepsilon(t)\varphi(x) \leq u(x, t)$ in $B_r(x_0) \times (0, t_0]$. In particular,

$$\mu_\varepsilon(t) \leq u(x_0, t), \quad t \in (0, t_0]. \quad (4.32)$$

Noting that $f \in RV_\rho$ and $\rho > p - 1$, in view of $\mu_\varepsilon(t) \rightarrow \infty$ as $t \rightarrow 0^+$, there exists t_n with $0 < t_n \leq t_0$ such that

$$\frac{\lambda_1 \mu_\varepsilon^{p-1}(t)}{(b_0 + \varepsilon)f(\mu_\varepsilon(t))} \leq \frac{1}{n}, \quad 0 < t \leq t_n.$$

Hence $\mu_\varepsilon(t)$ satisfies

$$\mu'_\varepsilon \geq -(b_0 + \varepsilon)(1 + 1/n)f(\mu_\varepsilon), \quad 0 < t \leq t_n; \quad \mu_\varepsilon(0) = \infty.$$

Using the arguments of Step 1 we can prove that there exist $\ell_n \nearrow 1$ and t_n^* with $0 < t_n^* \leq t_n$, such that $\ell_n \tilde{\mu}_\varepsilon(t) \leq \mu_\varepsilon(t)$, $\forall 0 < t < t_n^*$, $n \gg 1$. This combined with (4.32) gives $\ell_n \tilde{\mu}_\varepsilon(t) \leq u(x_0, t)$, $\forall 0 < t < t_n^*$, $n \gg 1$. Therefore, $\liminf_{t \rightarrow 0^+} \frac{u(x_0, t)}{\tilde{\mu}_\varepsilon(t)} \geq \ell_n$, $\forall n \gg 1$. Letting $n \rightarrow \infty$ we get (4.30).

Now we consider the case $2N/(N+2) < p < 2$. By [21, Theorems I and II], there is a constant $\lambda > 0$ such that the problem

$$\begin{cases} -\Delta_p \varphi = \lambda \varphi, & x \in B_r(x_0), \\ \varphi = 0, & x \in \partial B_r(x_0) \end{cases}$$

has a positive solution $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $0 < \varphi(x) < 1$ in $B_r(x_0) \setminus \{x_0\}$ and $\varphi(x_0) = 1$. Let η_ε be the unique positive solution of

$$\eta'_\varepsilon = -\lambda \eta_\varepsilon^{p-1} - (b_0 + \varepsilon)f(\eta_\varepsilon), \quad t > 0; \quad \eta_\varepsilon(0) = \infty.$$

Similar to the above we can prove that $\eta_\varepsilon(\sigma + t)\varphi(x) \leq u(x, t)$ in $B_r(x_0) \times (0, t_0]$ provided that $0 < \sigma \ll 1$. Letting $\sigma \rightarrow 0^+$ we deduce $\eta_\varepsilon(t)\varphi(x) \leq u(x, t)$ in $B_r(x_0) \times (0, t_0]$. In particular, $\eta_\varepsilon(t) \leq u(x_0, t)$ in $(0, t_0]$. By the same argument as above we obtain (4.30).

As above, choose $0 < \varepsilon_n \rightarrow 0^+$ and set $\varepsilon = \varepsilon_n$. Taking into account that $\lim_{t \rightarrow 0^+} \tilde{\mu}_{\varepsilon_n}(t) = \infty$, similar to the above arguments we can prove that there are $\sigma_n \rightarrow 0^+$ and $t'_n \rightarrow 0^+$ such that

$$(1 - \sigma_n)\tau(t) \leq \tilde{\mu}_{\varepsilon_n}(t), \quad \forall t \in (0, t'_n), \quad n \gg 1, \quad (4.33)$$

here $\tau(t)$ is the unique positive solution of (1.11). By (4.30) and (4.33), $\liminf_{t \rightarrow 0^+} \frac{u(x_0, t)}{\tau(t)} \geq 1 - \sigma_n$.

The limit (1.13) is obtained by taking $n \rightarrow \infty$. The proof is complete. \square

Proof of Theorem 1.3 Under our assumptions, it is easy to see that $f^*(u) = f(u)$, and hence $\xi^*(t) = \xi(t)$. By (1.9) and (1.10), there is a constant $l > 1$ such that $\underline{u}(x, t) \leq \bar{u}(x, t) \leq l\underline{u}(x, t)$ in Ω_T . The remainder of the proof is similar to that of [12, Theorem 1.4]. We omit the details. \square

A Appendix

A.1 Some basic results of regular variation theory

In this subsection, we gather some basic results of regular variation theory that are needed in this paper. In most cases, we refer the reader to the basic references (such as [4]) and omit the proofs. However, in certain instances, we feel that we need to provide the proofs as they are not readily available in the literature.

Proposition A.1 (*Representation Theorem*) *The function $L(u)$ is slowly varying at infinity if and only if it can be written as*

$$L(u) = M(u) \exp \left\{ \int_b^u \frac{\varphi(t)}{t} dt \right\}, \quad \forall u \geq b$$

for some $b > 0$, where the function $\varphi \in C([b, \infty))$ satisfies $\lim_{u \rightarrow \infty} \varphi(u) = 0$ and $M(u)$ is measurable on $[b, \infty)$ with $\lim_{u \rightarrow \infty} M(u) = M^* \in (0, \infty)$.

Definition A.1 *A function $\hat{L}(u)$ is referred to as normalized slowly varying at infinity if it satisfies the requirements in Proposition A.1 with $M(u)$ replaced by M^* . The function $R(u) = M^* u^\rho \hat{L}(u)$ is called normalized regularly varying at infinity of index ρ and we write $R(u) \in NRV_\rho$.*

We say that $R(u)$ is regularly varying at the origin (from the right) of index $\rho \in \mathbb{R}$, denoted by $R \in RVZ_\rho$, if $R(1/u) \in RV_{-\rho}$. The set of all normalized regularly varying functions at the origin of index ρ is denoted by $NRVZ_\rho$.

Similar to [7, Remark 2.2], we have

Lemma A.1 *Let $k \in \mathcal{K}_\ell$ with $\ell > 0$. Then $k(1/u)$ belongs to $NRV_{(\ell-1)/\ell}$. Furthermore, we have $K(1/u)$ belongs to $NRV_{-1/\ell}$. And hence, $K(u) \in NRVZ_{1/\ell}$ and $k(u) \in NRVZ_{(1-\ell)/\ell}$.*

Lemma A.2 ([6, Lemma 2.3]) *Suppose that the condition (F_1) holds and the function ϕ is given by (1.7). Then*

- (i) $-\phi'(t) = (p' F(\phi(t)))^{1/p}$, and $|\phi'(t)|^{p-2} \phi''(t) = \frac{p'}{p} f(\phi(t))$, where $p' = \frac{p}{p-1}$;
- (ii) $-\phi' \in NRVZ_{-r}$, $\phi \in NRVZ_{1-r}$, where $r = (\rho + 1)/(\rho + 1 - p) > 1$.

Following the discussion of [7, Section 2], we can prove

Lemma A.3 *A function $f \in NRV_\rho$ (or $f \in NRVZ_\rho$) if and only if $f \in C^1[a_1, \infty)$ (or $f \in C^1(0, a_1)$) for some $a_1 > 0$ and $\lim_{s \rightarrow \infty} \frac{sf'(s)}{f(s)} = \rho$ (or $\lim_{s \rightarrow 0^+} \frac{sf'(s)}{f(s)} = \rho$).*

In view of Lemma A.3 we can prove

Lemma A.4 *Assume that $f \in NRV_\rho$ (or $f \in NRVZ_\rho$) and $\rho \neq 0$. If $f'(s) > 0$, then the inverse function $f^{-1}(y)$ of $f(s)$ belongs to $NRV_{1/\rho}$ (or $f^{-1}(y) \in NRVZ_{1/\rho}$). If $f'(s) < 0$, then the inverse function $f^{-1}(y)$ of $f(s)$ belongs to $NRVZ_{1/\rho}$ (or $f^{-1}(y) \in NRV_{1/\rho}$).*

Since $\phi \in NRVZ_{1-r}$ and $r > 1$, $K \in RVZ_{1/\ell}$, by Lemma A.4 we have $\phi^{-1} \in RV_{1/(1-r)}$, $K^{-1} \in RVZ_\ell$. Thanks to $k \in RVZ_{(1-\ell)/\ell}$, it follows that

$$k \circ K^{-1} \circ \phi^{-1}(s) \in RV_{\frac{1-\ell}{1-r}}, \quad (\text{A.1})$$

$$f^*(s) = (k \circ K^{-1} \circ \phi^{-1}(s))^p f(s) \in RV_{\frac{p(1-\ell)}{1-r} + \rho} = RV_q. \quad (\text{A.2})$$

In the following, C represents a generic positive constant which can differ from line to line.

Lemma A.5 *Let $\varrho > 0$ be a constant. Assume that f is a positive continuous function in $(0, \infty)$, and $f \in RV_\gamma$ with $\gamma > \max\{1, \varrho\}$. Let $w(x, t)$ be a positive function with positive lower bound w_0 . Then for any given constant $C \geq 1$, there is a constant $\Lambda > 0$, which depends on w_0 , C and γ , such that, for all (x, t) ,*

$$C(\Lambda + \Lambda^\varrho) f(w(x, t)) < f(\Lambda w(x, t)).$$

Proof. Denote $\alpha = \max\{1, \varrho\}$ and choose $\sigma > 0$ satisfying $\gamma - \sigma > \alpha$. Choose $\Lambda_0 \geq 2$ and $0 < \varepsilon \ll 1$ with $(1 - \varepsilon)\Lambda_0^\sigma > 2C$. By the definition, there is a constant $z_0 = z_0(\Lambda_0) > 0$ such that

$$f(\Lambda_0 z) \geq (1 - \varepsilon)\Lambda_0^\gamma f(z), \quad \forall z \geq z_0. \quad (\text{A.3})$$

For any positive integer $j \geq 2$ and $z \geq z_0$, we have $\Lambda_0^{j-i} z \geq z \geq z_0$ for all $1 \leq i \leq j - 1$, and by inductively

$$\begin{aligned} f(\Lambda_0^j z) &= f(\Lambda_0 \Lambda_0^{j-1} z) \geq (1 - \varepsilon)\Lambda_0^\gamma f(\Lambda_0^{j-1} z) \geq \dots \\ &\geq [(1 - \varepsilon)\Lambda_0^\gamma]^j f(z) = [(1 - \varepsilon)\Lambda_0^\sigma]^j f(z) \Lambda_0^{(\gamma - \sigma)j} \\ &> 2C\Lambda_0^{(\gamma - \sigma)j} f(z). \end{aligned} \quad (\text{A.4})$$

As $f(z) \rightarrow \infty$ as $z \rightarrow \infty$, we can choose z_0 so large that $f(z) \leq f(z_0)$ for all $w_0 \leq z \leq z_0$.

Since $f \in RV_\gamma$, we have $\lim_{s \rightarrow \infty} s^{-\gamma + \sigma} f(s) = \infty$. For the given constant $A > 2Cf(z_0)/w_0^{\gamma - \sigma}$, there is a constant $S_0 > 1$ such that $f(s) \geq As^{\gamma - \sigma}$ for all $s \geq S_0$. It is obvious that there is a constant $\Lambda^* > 1$ such that $\Lambda z > S_0$ for all $\Lambda \geq \Lambda^*$ and $w_0 \leq z \leq z_0$. Therefore,

$$f(\Lambda z) \geq A(\Lambda z)^{\gamma - \sigma} \geq Aw_0^{\gamma - \sigma} \Lambda^{\gamma - \sigma} \frac{f(z)}{f(z_0)} \geq 2C\Lambda^{\gamma - \sigma} f(z), \quad \forall \Lambda \geq \Lambda^*, \quad w_0 \leq z \leq z_0. \quad (\text{A.5})$$

Note that $\Lambda_0 \geq 2$, we can choose an integer $j \geq 2$ such that $\Lambda_0^j \geq \Lambda^*$. Take $\Lambda = \Lambda_0^j$, then our conclusion is true. In fact, for those (x, t) with $w(x, t) \leq z_0$, by (A.5)

$$f(\Lambda w(x, t)) > 2C\Lambda^{\gamma - \sigma} f(w(x, t)) > 2C\Lambda^\alpha f(w(x, t)).$$

For those (x, t) with $w(x, t) > z_0$, by (A.4),

$$\begin{aligned} f(\Lambda w(x, t)) &= f(\Lambda_0^j w(x, t)) > 2C\Lambda_0^{(\gamma - \sigma)j} f(w(x, t)) \\ &= 2C\Lambda^{\gamma - \sigma} f(w(x, t)) > 2C\Lambda^\alpha f(w(x, t)). \end{aligned}$$

The proof is complete. \square

When f is increasing, the following better result can be obtained:

Lemma A.6 *Under the conditions of Lemma A.5, we further assume that f is increasing in $(0, \infty)$. Then for any given constant $C \geq 1$, there is a constant $\Lambda^* > 0$, which depends on w_0 , C and γ , such that for all (x, t) and all $\Lambda \geq \Lambda^*$,*

$$C(\Lambda + \Lambda^\varrho) f(w(x, t)) < f(\Lambda w(x, t)).$$

Lemma A.7 *Suppose that $f \in RV_\gamma$ with $\gamma \in \mathbb{R}$, is continuous, increasing and positive in $(0, \infty)$. Then for any given $\tau > 0$, there is a constant $c = c(\tau) > 0$, such that*

$$f(a) + f(b) > cf(a + b), \quad \forall a, b \geq \tau.$$

Proof. We assume by contradiction that there exist two sequences $\{a_n\}$ and $\{b_n\}$, such that $f(a_n) + f(b_n) \leq \frac{1}{n}f(a_n + b_n)$. Since f is continuous, increasing and positive in $[\tau, \infty)$, it follows $a_n + b_n \rightarrow \infty$ as $n \rightarrow \infty$. Suppose that $a_n \leq b_n$ without loss of generality. Since f is increasing on $(0, \infty)$, we have $f(b_n) \leq \frac{1}{n}f(2b_n)$, i.e., $f(2b_n)/f(b_n) \geq n$. On the other hand, since $f \in RV_\gamma$, we have $\lim_{n \rightarrow \infty} [f(2b_n)/f(b_n)] = 2^\gamma$, which is a contradiction. \square

Lemma A.8 Assume that f is a positive continuous function and $f \in RV_\gamma$ with $\gamma > 0$. Let $\tau > 0$ be a given constant. Then there exist a positive, continuous and increasing function $g \in RV_\gamma$ and a constant σ , such that

$$\sigma g(u) \leq f(u) \leq g(u), \quad \forall u \geq \tau.$$

Proof. As $u^{-\gamma}f(u)$ is a slow variation function, by Proposition A.1,

$$u^{-\gamma}f(u) = M(u)\psi(u), \quad \text{with } \psi(u) = \exp \left\{ \int_b^u \frac{\varphi(t)}{t} dt \right\}, \quad u \geq b > 0.$$

Since $\lim_{u \rightarrow \infty} M(u) = M^* \in (0, \infty)$, there is a constant $u_1 > 0$ such that $M^*/2 < M(u) < 2M^*$ for all $u \geq u_1$. Hence,

$$\frac{1}{2}M^*u^\gamma\psi(u) \leq f(u) \leq 2M^*u^\gamma\psi(u), \quad \forall u \geq u_1.$$

The direct computation gives $(u^\gamma\psi(u))' = u^{\gamma-1}[\gamma + \varphi(u)]\psi(u)$. By use of $\lim_{u \rightarrow \infty} \varphi(u) = 0$, it follows that there is a $u_2 > 0$ such that $\gamma + \varphi(u) > 0$ when $u \geq u_2$. That is, the function $u^\gamma\psi(u)$ is increasing for $u \geq u_2$. Take a positive, continuous and increasing function $g_1(u)$ such that $g_1(u) = u^\gamma\psi(u)$ when $u \geq u_0 = \max\{u_1, u_2\}$. It is obvious that $g_1(u) \in RV_\gamma$ and $\frac{1}{2}M^*g_1(u) \leq f(u) \leq 2M^*g_1(u)$ for all $u \geq u_0$. Note that both f and g_1 are continuous and positive in $[\tau, u_0]$, there is a constant $C > 0$ such that $C^{-1}g_1(u) \leq f(u) \leq Cg_1(u)$ for all $\tau \leq u \leq u_0$. Take $g(u) = (2M^* + C)g_1(u)$, then our conclusion holds. \square

A.2 Some results on the unique solution of (1.5)

Lemma A.9 Assume that $g(u)$ and $h(u)$ are continuous functions and $h(u)$ is positive in $[a, \infty)$ for some constant $a > 0$, and that $h(u) \in NRV_\gamma$ with $\gamma > 1$. Let $v(t)$ and $w(t)$ be the positive solutions of the problems, respectively:

$$\begin{aligned} v'(t) &= -g(v(t)), \quad t > 0; & v(0) &= \infty, \\ w'(t) &= -h(w(t)), \quad t > 0; & w(0) &= \infty. \end{aligned}$$

If $\lim_{u \rightarrow \infty} \frac{g(u)}{h(u)} = 1$, then $\lim_{t \rightarrow 0^+} \frac{v(t)}{w(t)} = 1$.

Proof. Note that $h(u) \in NRV_\gamma$ and $\gamma > 1 + \nu$, similar to the proof of Lemma A.8 we have that $h(u)/u^{1+\nu}$ is increasing when $u \geq u_0$ for some large constant u_0 . Choose ε_n with $0 < \varepsilon_n < 1/2$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. In view of $g(u)/h(u) \rightarrow 1$ as $u \rightarrow \infty$, there is $u_n \geq 2^{1/\nu}u_0$ such that $(1 - \varepsilon_n)h(u) \leq g(u) \leq (1 + \varepsilon_n)h(u)$ when $u \geq u_n$.

Thanks to the fact that $v(t), w(t) \rightarrow \infty$ as $t \rightarrow 0$, there is $t_n > 0$ such that $v(t) \geq u_n$ for all $0 < t \leq t_n$. Therefore,

$$(1 - \varepsilon_n)h(v(t)) \leq g(v(t)) \leq (1 + \varepsilon_n)h(v(t)), \quad \forall 0 < t \leq t_n.$$

Hence

$$v'(t) \leq -(1 - \varepsilon_n)h(v(t)), \quad v(t) \geq 2^{1/\nu}u_0, \quad \forall 0 < t \leq t_n; \quad v(0) = 0.$$

Denote $\sigma_n = (1 - \varepsilon_n)^{1/\nu}$. In view of $\varepsilon_n < 1/2$, we see that $\sigma_n v(t) \geq u_0$ for all n and $0 < t \leq t_n$. Notice that $h(u)/u^{1+\nu}$ is increasing for $u \geq u_0$ and $\sigma_n < 1$, it is easily seen that $\sigma_n(1 - \varepsilon_n)h(v(t)) = \sigma_n^{1+\nu}h(v(t)) \geq h(\sigma_n v(t))$ in $(0, t_n]$. Therefore, $y_n(t) = \sigma_n v(t)$ satisfies

$$y_n'(t) \leq -\sigma_n(1 - \varepsilon_n)h(v(t)) \leq -h(y_n(t)), \quad \forall 0 < t \leq t_n.$$

The comparison principle gives that $y_n(t) = \sigma_n v(t) \leq w(t)$ for all $0 < t \leq t_n$. Hence $\lim_{t \rightarrow 0^+} \frac{v(t)}{w(t)} \leq 1/\sigma_n$, and consequently $\lim_{t \rightarrow 0^+} \frac{v(t)}{w(t)} \leq 1$ by letting $n \rightarrow \infty$.

Similarly, we can prove that $\lim_{t \rightarrow 0^+} \frac{v(t)}{w(t)} \geq 1$. \square

The following corollary can be drawn; we shall omit the proof:

Corollary A.1 *In Lemma A.9, if we replace the condition $h(u) \in NRV_\gamma$ by $h(u) \in RV_\gamma$ with $\gamma > 1$, the conclusion is also true.*

Following the same line of argument, we can also establish:

Lemma A.10 *Assume that $g(u)$ and $h(u)$ are continuous functions and $h(u)$ is positive in $[a, \infty)$ for some constant $a > 0$. Suppose further that $h(u) \in RV_\gamma$ with $\gamma > 1$. Let $v(t)$ and $w(t)$ be the positive solutions of the problems, respectively:*

$$\begin{aligned} v'(t) &= -g(v(t)), \quad t > 0; & v(0) &= \infty, \\ w'(t) &= -h(w(t)), \quad t > 0; & w(0) &= \infty. \end{aligned}$$

If $\lim_{u \rightarrow \infty} \frac{g(u)}{h(u)} = c$ for some constant $c > 0$, then for any $\nu : 0 < \nu < \gamma - 1$,

$$\begin{aligned} c^{1/\nu} &\leq \liminf_{t \rightarrow 0^+} \frac{w(t)}{v(t)} \leq \limsup_{t \rightarrow 0^+} \frac{w(t)}{v(t)} \leq 1 \quad \text{if } c \leq 1, \\ 1 &\leq \liminf_{t \rightarrow 0^+} \frac{w(t)}{v(t)} \leq \limsup_{t \rightarrow 0^+} \frac{w(t)}{v(t)} \leq c^{1/\nu} \quad \text{if } c > 1. \end{aligned}$$

Using Lemma A.10, we can also obtain:

Lemma A.11 *Assume that $g(u)$ and $h(u)$ are positive continuous differentiable functions, and $g(u) \in RV_\theta$, $h(u) \in RV_\gamma$ with $\gamma + \theta > 1$. Let $c > 0$ be a given constant. Denote by $v(t)$ and $w(t)$ solutions of the following problems, respectively:*

$$\begin{aligned} v'(t) &= -g(cv(t))h(v(t)), \quad t > 0; & v(0) &= \infty, \\ w'(t) &= -g(w(t))h(w(t)), \quad t > 0; & w(0) &= \infty. \end{aligned}$$

Then there exists a constant $C > 1$ such that $C^{-1}v(t) \leq w(t) \leq Cv(t)$ in $(0, T]$.

A.3 Some results on the corresponding elliptic boundary blow-up problem

In this final subsection, we recall, for the sake of ease of reference for the reader, some results about the boundary blow-up solutions of the p -Laplacian elliptic equation

$$\begin{cases} \Delta_p u = b(x)f(u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \quad (\text{A.6})$$

where $b(x) \in C^\alpha(\Omega)$ for some $0 < \alpha < 1$ with $b(x) \geq 0$ and $b(x) \not\equiv 0$ in Ω . Set $\Omega_0 = \{x \in \Omega : b(x) = 0\}$ and assume that $\bar{\Omega}_0 \subset \Omega$ and $b(x) > 0$ in $\Omega \setminus \bar{\Omega}_0$.

Theorem A.1 ([6, Theorem 1.1]) *Assume that f satisfies (F_2) and*

$$(F_3) \quad \int_1^\infty F^{-1/p}(t)dt < \infty.$$

Then the problem (A.6) has at least one positive solution.

In fact, (F_1) implies (F_3) , see [6, Lemmas 2.1 and 2.2, and Remark 2.4].

Theorem A.2 ([6, Theorem 1.2]) *Assume that $(F_1) - (F_2)$ hold, the function ϕ is defined by (1.7).*

(i) *If there exist a function $k \in \mathcal{K}_\ell$ and a positive continuous function $\beta(y)$ defined on $\partial\Omega$ such that*

$$\lim_{\Omega \ni x \rightarrow y} \frac{b(x)}{k^p(d(x))} = \beta(y) \quad \text{uniformly for } y \in \partial\Omega,$$

then (A.6) has a unique positive solution and the blow-up rate is given by

$$\lim_{\Omega \ni x \rightarrow y} \frac{u(x)}{\phi(K(d(x)))} = \left(\frac{r + \ell - 1}{r\beta(y)} \right)^{\frac{r-1}{p}} \quad \text{uniformly for } y \in \partial\Omega.$$

(ii) *Suppose that there exist a function $k \in \mathcal{K}_\ell$ and constants $0 < \beta_1 \leq \beta_2$, $\delta > 0$, such that*

$$\beta_1 k^p(d(x)) \leq b(x) \leq \beta_2 k^p(d(x)) \quad \text{for all } x \in \Omega \text{ with } d(x) \leq \delta.$$

Then the problem (A.6) has a positive solution u satisfying

$$\liminf_{d(x) \rightarrow 0} \frac{u(x)}{\phi(K(d(x)))} \geq \left(\frac{r + \ell - 1}{r\beta_2} \right)^{\frac{r-1}{p}}, \quad \limsup_{d(x) \rightarrow 0} \frac{u(x)}{\phi(K(d(x)))} \leq \left(\frac{r + \ell - 1}{r\beta_1} \right)^{\frac{r-1}{p}}.$$

(iii) *When $p = 2$ and $\ell \neq 0$, under the condition of (ii), the positive solution of (A.6) is also unique.*

Remark A.1 *From the proof of [6, Theorem 1.2] it can be seen that, for any given $y \in \partial\Omega$, if the limit*

$$\lim_{\Omega \ni x \rightarrow y} \frac{b(x)}{k^p(d(x))} = \beta(y)$$

exists, then any positive solution $u(x)$ of (A.6) satisfies

$$\lim_{\Omega \ni x \rightarrow y} \frac{u(x)}{\phi(K(d(x)))} = \left(\frac{r + \ell - 1}{r\beta(y)} \right)^{\frac{r-1}{p}}.$$

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